Math 71 Fall 2015
Homework #4: rings and ideals

(1) Let A be a commutative ring with identity and $A[X]$ the ring of polynomials with coefficients in A.

(a) Prove that (X) is a prime ideal of $A[X]$ if and only if A is an integral domain.

(b) Prove that (X) is maximal if and only if A is a field.

(c) Let I and J be ideals in A and P a prime ideal such that $IJ \subset P$. Prove that either I or J is contained in P.

(2) Let A be a commutative ring with identity and consider a polynomial $P \in A[X]$ of degree $n \geq 1$ with leading coefficient 1.

(a) Verify that every class in $A[X]/(P)$ has a representative in $A[X]$ of degree at most $n-1$.

(b) Let $U, V \in A_{n-1}[X]$ be distinct polynomials. Prove that their images in $A[X]/(P)$ are distinct.

(c) Assume that P can be factored in $A[X]$. Prove that $A[X]/(P)$ has zero divisors.

(d) Assume that $P = X^n - a$ where $a \in A$ is nilpotent[1]. Prove that the image of X in $A[X]/(P)$ is nilpotent.

(3) Consider the ring $\mathbb{Z}[i] = \{a + ib \, , \; a, b \in \mathbb{Z}\}$.

(a) Prove that every point in the complex plane is at distance strictly less than 1 from an element in $\mathbb{Z}[i]$.

(b) Prove that $\mathbb{Z}[i]$ is a Euclidean domain and determine $\mathbb{Z}[i]^\times$.

[1] An element a in a ring is said *nilpotent* if $a^m = 0$ for some positive integer m.

(4) Let A be a ring with identity and consider the map $\varphi : \mathbb{N} \longrightarrow A$ defined by $\varphi(n) = \underbrace{1 + \ldots + 1}_{n \text{ times}}$.

(a) Prove that φ extends uniquely to a ring homomorphism from $\mathbb{Z}$ to A.

(b) Prove the existence of a non-negative integer κ such that $\ker \varphi = \kappa\mathbb{Z}$.

The integer κ of the previous question is called the *characteristic* of A.

(c) Prove that the characteristic of an integral domain is either 0 or a prime number.

(d) Determine the characteristics of $\mathbb{Q}$, $\mathbb{Z}[X]$ and $\mathbb{Z}/n\mathbb{Z}[X]$.

(e) Let p be a prime number and A a commutative ring of characteristic p. Prove that $(a+b)^p = a^p + b^p$ in A[2].

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[2]Be warned: mentioning this result to calculus students may constitute a violation of the Honor Code.