MATH 71 - ABSTRACT ALGEBRA
FALL 2015
MIDTERM 1

DURATION: 2 HOURS

This exam consists of 4 independent problems. Treat them in the order of your choosing, starting each problem on a new page.

Every claim you make must be **fully justified or quoted** as a result studied in class.

Problem 1

1. Let G be a group and H a subgroup. The relation defined on G by
$$x \sim y \Leftrightarrow x^{-1}y \in H$$
is an equivalence relation. For g element of G, describe the class of g.

2. Is $\mathfrak{S}_3$ cyclic?

3. Find the order of $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 12 & 4 & 1 & 6 & 8 & 11 & 10 & 5 & 7 & 9 & 2 & 3 \end{pmatrix}$ in $\mathfrak{S}_{12}$.

4. Give two non-isomorphic groups of cardinal 8. Explain why they are not isomorphic.

5. Let G, G' be groups and $\varphi \in \mathrm{Hom}(G, G')$. Assume that G is cyclic and φ is surjective. Prove that G' is cyclic.

6. Determine the subgroup of $(\mathbb{Q}_+^\times, \times)$ generated by $A = \left\{\frac{1}{p}\ ,\ p \text{ prime}\right\}$.

Problem 2

Recall that $D_{2n} = \langle r, s \mid r^n = 1\ ,\ s^2 = 1\ ,\ rs = sr^{-1} \rangle$.

1. Show that every element of D_{2n} that is not a power of r has order 2.

2. Deduce that every element of D_{2n} is the product of elements of order 2.

3. Let G be a finite group generated by distinct elements a and b, both of order 2. Prove that G is isomorphic to D_{2n}, where $n = |ab|$.

Hint: prove that $\langle a, b \rangle = \langle a, ab \rangle$

Problem 3

1. Let G_1 and G_2 be groups and consider the product $G = G_1 \times G_2$, with the group law

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 y_1, x_2 y_2).$$

Prove that $G_1 \times \{1_{G_2}\} \lhd G$ and that the quotient $G/G_1 \times \{1_{G_2}\}$ is isomorphic to G_2.

2. Let F be a field. Consider the following subgroups of $\mathrm{SL}(2, F)$:

$$P = \left\{ \begin{bmatrix} a & t \\ 0 & a^{-1} \end{bmatrix}\ ,\ a \in F^\times\ ,\ t \in F \right\} \qquad , \qquad N = \left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\ ,\ t \in F \right\}.$$

Prove that $N \lhd P$ and that P/N is isomorphic to $F^\times$.

Problem 4

Recall that $\mathrm{Aut}(G)$ denotes the set of isomorphisms of a group G onto itself, and that it is a group under composition. Let

$$\mathrm{Inn}(G) = \{c_g\ ,\ g \in G\}\ ,\ \text{where} \quad c_g(x) = gxg^{-1} \text{ for all } x \in G.$$

1. Verify that $c_g \in \mathrm{Aut}(G)$ for all $g \in G$.

2. Prove that $\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Aut}(G)$.

3. Is $\mathrm{Inn}(G)$ normal in $\mathrm{Aut}(G)$?

4. Prove that $\mathrm{Inn}(G)$ is isomorphic to a quotient of G.