MATH 71 - ABSTRACT ALGEBRA
FALL 2015
MIDTERM 2

DURATION: 2 HOURS

This examination consists of four independent problems. Treat them in the order of your choosing, starting each problem on a new page. **Every claim** you make must be **fully justified or quoted** as a result studied in class.

Problem 1

The questions in this problem are independent.

1. Let G be a group acting on a finite set X and $x_1, \ldots, x_n$ a family of representatives of all the orbits. Prove that $\#X = \sum_{i=1}^{n} [G : \mathrm{Stab}_G(x_i)]$.

2. Let A be a ring. Prove that $0 \times a = 0$ for all $a \in A$.

3. Consider the ring A of functions from $\mathbb{R}$ to $\mathbb{R}$. For each of the following subsets, determine if it is a subring A. If so, determine if it is an ideal.

(a) $A_{-3} = \{f : \mathbb{R} \longrightarrow \mathbb{R}\ ,\ f(-3) = 0\}$

(b) $E = \{\text{even functions in } A\}$

(c) $O = \{\text{odd functions in } A\}$

Problem 2

1. Let F be a field. For $a \in F^\times$ let $\varphi(a)$ denote the map $\begin{array}{ccc} F & \longrightarrow & F \\ t & \longmapsto & a^2 t \end{array}$.

(a) Prove that $\varphi(a)$ is an automorphism of the additive group $(F, +)$.

(b) Prove that φ is a group homomorphism from $F^\times$ to $\mathrm{Aut}(F)$.

2. Prove that $F \rtimes_\varphi F^\times$ is isomorphic to the subgroup $P = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}\ ,\ a \in F^\times\ ,\ b \in F \right\}$ of $\mathrm{SL}(2, F)$.

Problem 3

Let $n \geq 3$. Recall that the alternating group $\mathfrak{A}_n$ is generated by the 3-cycles. A *square* in a group G is an element of the form $g^2 = g \cdot g$.

1. Prove that every 3-cycle σ in $\mathfrak{S}_n$ is a square. *Hint: compute* σ^4.

2. Let H be the subgroup of $\mathfrak{S}_n$ generated by the squares. Prove that $H = \mathfrak{A}_n$.

3. Let N be a subgroup of index 2 in $\mathfrak{S}_n$

(a) Prove that $\sigma^2 \in N$ for any $\sigma \in \mathfrak{S}_n$.

(b) Deduce that $N = \mathfrak{A}_n$.

Problem 4

Let K be the subgroup of $\mathrm{SL}(2,\mathbb{R})$ consisting of matrices of the form $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ with $c^2 + s^2 = 1$ and

$$\mathfrak{g} = \left\{ X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} , \ \mathrm{Tr}(X) = a + d = 0 \right\}.$$

Note that $\mathfrak{g}$ is *not* a subgroup of $\mathrm{SL}(2,\mathbb{R})$.

1. Verify that K acts on $\mathfrak{g}$ by $k \cdot X = kXk^{-1}$ where $k \in K$ and $X \in \mathfrak{g}$.

Let $\mathfrak{a} = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix} , \ \alpha \in \mathbb{R} \right\} \subset \mathfrak{g}$. Again, $\mathfrak{a}$ is not a subgroup of $\mathrm{SL}(2,\mathbb{R})$.

2. Determine all the elements of the subgroup $N = \{k \in K \, , \ k\,\mathfrak{a}\,k^{-1} \subset \mathfrak{a}\}$ of K.

3. Determine the subgroup $C = \{k \in K \, , \ kXk^{-1} = X \text{ for all } X \in \mathfrak{a}\}$ of N.

4. Prove that N is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ and that N/C is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.