MATH 71 - ABSTRACT ALGEBRA
FALL 2015
MIDTERM 2 - TAKE-HOME

DUE OCTOBER 30

Problem 1

The goal of this problem is to determine for which values of n there exists a unique group of order n up to isomorphism. In what follows, n is a positive integer with prime decomposition $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$.

1. (a) Assume n prime. Determine, up to isomorphism, all the groups of order n.

(b) Prove that if $\alpha_i \geq 2$ for some $i \in \{1, \ldots, s\}$, then there are at least two non-isomorphic groups of order n. (Think about Klein's group V_4.)

From now on, we assume that $\alpha_i = 1$ for all $i \in \{1, \ldots, s\}$.

2. Recall (without proof) the expression of the Euler Indicator $\varphi(n)$ in that case.

3. Let p and q be distinct prime numbers such that $p|(q-1)$.

(a) Prove the existence of a non-abelian group of order pq.

(b) Deduce that if all groups of order n are isomorphic, then $n \wedge \varphi(n) = 1$.

We shall prove the converse by contradiction. Let n be the smallest integer for which $n \wedge \varphi(n) = 1$ and there exists a group G of order n that is not isomorphic to $\mathbb{Z}/n\mathbb{Z}$ (assuming the existence of such integers).

4. (a) Prove that $m \wedge \varphi(m) = 1$ for any divisor m of n.

(b) Prove that every proper subgroup and every non-trivial quotient group of G is cyclic.

(c) Deduce that the center of G is trivial. (Hint: consider $G/Z(G)$ and use the Fundamental Theorem.)

A *maximal subgroup* of a group Γ is a proper subgroup H such that the only subgroups of Γ containing H are H and Γ.

5. Let U be a maximal subgroup of G and $x \neq 1$ in U.

(a) Prove that $U = C_G(x)$.

(b) Deduce that any two distinct maximal subgroups of G have trivial intersection.

We admit the following result: every maximal subgroup of G is equal to its own normalizer:

$$U = N_G(U).$$

6. Let U be a maximal subgroup, u its order and $\mathfrak{U}$ the union of all conjugates of U in G.

(a) Determine the number of conjugates of U and the order of each such conjugate.

(b) Verify that the conjugates of U are maximal and deduce that $\mathfrak{U}$ contains $n - \frac{n}{u}$ elements different from the identity.

7. Let $x \in G \setminus \mathfrak{U}$. Consider V a maximal subgroup of G containing x. Denote by v its order and by $\mathfrak{V}$ the union of all conjugates of V.

(a) Prove that $\mathfrak{U} \cup \mathfrak{V}$ contains $2n - \frac{n}{u} - \frac{n}{v}$ elements different from 1.

(b) Compare to the cardinality of $G \setminus \{1\}$ and deduce a contradiction.

8. Conclude.

Problem 2

Let F be a field and consider the groups $G = \mathrm{SL}(2, F)$ and $N = \left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, t \in F \right\}$.
Let X_2 denote the set $F^2 \setminus \{(0,0)\}$, with elements written as column matrices.

1. Is N is normal in G?

2. Prove that G acts on X_2 *via* left matrix multiplication.

For $g \in G$, let $c_1(g)$ denote the first column of g.

3. Prove that the map $\varphi : \begin{array}{ccc} G/N & \longrightarrow & \mathsf{X}_2 \\ gN & \longmapsto & c_1(g) \end{array}$ is well-defined.

4. Prove that φ is a G-equivariant bijection.

From now on, assume $n \geq 1$ and let $G = \mathrm{SL}(n+1, F)$, while Y_{n+1} denotes the set of F-valued matrices with $n+1$ rows and n columns.

5. Verify that G acts on Y_{n+1} *via* left matrix multiplication.

6. Let $x_0 = \begin{bmatrix} I_n \\ \hline 0 \ \ldots \ 0 \end{bmatrix}$. Determine the group $N = \mathrm{Stab}_G(x_0)$.

7. Describe the map $b : \begin{array}{ccc} G & \longrightarrow & \mathsf{Y}_{n+1} \\ g & \longmapsto & g \cdot x_0 \end{array}$.

8. Prove that G/N is in G-equivariant bijection with the subset X_{n+1} of Y_{n+1} consisting of the elements of rank n.