Math 71

Homework 9 — practice problems

p301: #1 Let F be a field, and $f(x) \in F[x]$ a polynomial of degree $n \geq 1$. Let $g \mapsto \overline{g}$ denote the reduction homomorphism $F[x] \to F[x]/(f)$. Prove that for each $\overline{g}$ there exists a unique polynomial $g_0 \in F[x]$ with degree $\leq n-1$ so that $\overline{g} = \overline{g_0}$. Show that this means that $F[x]/(f)$ is a vector space over F with basis $\{\overline{1}, \overline{x}, \ldots, \overline{x^{n-1}}\}$.

p301/311 Let F be a finite field with q elements, and $f \in F[x]$ of degree $n \geq 1$.

(a) Show that $F[x]/(f)$ is a ring with q^n elements.

(b) Show that $F[x]/(f)$ is a field with q^n elements iff f is irreducible in $F[x]$.

(c) Use this idea to construct fields with 9 and 49 elements.

p301: #4 Let F be a finite field. Show that $F[x]$ contains infinitely many primes.

pXXX: #n Find all ideals in $\mathbb{Q}[x]$ which contain $(x^4 - 1)$. Identify those which are prime, maximal.

p311: # 1 Irreducibility of polynomials. Determine whether or not the polynomials below are irreducible. If not, factor them as a product of irreducibles. $\mathbb{F}_p$ is the field with p elements.

(a) $x^2 + x + 1$ in $\mathbb{F}_2[x]$.

(b) $x^3 + x + 1$ in $\mathbb{F}_3[x]$.

(c) $x^4 + 1 = x^4 - 4$ in $\mathbb{F}_5[x]$

(d) $x^4 + 10x^2 + 1$ in $\mathbb{Q}[x]$

p311: #3 Show that the polynomial $(x-1)(x-2)\cdots(x-n)-1$ is irreducible over $\mathbb{Z}$ for all $n \geq 1$. Hint: If the polynomial factors, consider the values of the factors at $x = 1, 2, \ldots, n$.

p311: #11 Prove that $x^2 + y^2 - 1$ is irreducible in $\mathbb{Q}[x, y]$.