Math 74 Spring 2005

Topology II: Introduction to Algebraic Topology

Final Exam (take-home)

due at 2pm on Tuesday June 7
in the office of **Prof. Greg Leibon**, 308 Bradley Hall

Your name (please print): ______________________________

Instructions: This is an open book, open notes exam. You can use any printed matter (or your class notes) you like but you **can not** consult one another or other humans. **Use of calculators is not permitted**. You must justify all of your answers to receive credit.

The exam should be submitted **no later than 2pm on Tuesday June 7 to Prof. Greg Leibon** in 308 Bradley Hall (3rd floor). If he is not in his office when you submit your exam, then please **write the time** you finished working on it and **slide it under the door** of his office.

You also should

- Write/print on **one side of paper only** (no duplex printing). Add additional sheets of paper, if necessary;
- **Don't staple you exam**, but use a paper clip or another easily removable holder of a similar nature.

You can receive two (2) bonus points for following each of the two rules above correctly.

The exam total score is the sum of your **6** (out of **7**) best scores plus the bonus points.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Grader's use only

1.	________	/21
2.	________	/21
3.	________	/21
4.	________	/21
5.	________	/21
6.	________	/21
7.	________	/21
Bonus points:	________	/4
Total:	________	/130

1. Construct infinitely many non-homotopic retractions $S^1 \vee S^1$ onto S^1. Don't forget to prove that they are indeed pairwise non-homotopic.

2. Let X be a torus $S^1 \times S^1$ with two disks attached along the circles $S^1 \times \{x_0\}$ and $\{x_0\} \times S^1$. Prove that X is homotopy equivalent to the sphere S^2.

3. (two-dimensional Borsuk-Ulam Theorem) Let $f : S^2 \to \mathbb{R}^2$ be a continuous map. Prove that there exists $x \in S^2$ such that $f(x) = f(-x)$.
Hint: use a proof by contradiction. If such a map existed, it would induce a continuous map $\varphi : \mathbb{R}P^2 \to \mathbb{R}P^1$, such that $\varphi_* : \pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^1)$ is non-trivial. Why is it impossible? Don't forget to justify **all** your steps.

4. Let $L \subset \mathbb{R}^3$ be the union of n (distinct) lines through the origin and $X = \mathbb{R}^3 \setminus L$. Compute $\pi_1(X)$.
Hint: Find a (strong) deformation retract of X that is easy to deal with. You might want to start with $n = 1, 2, 3$.

5. Let X be a path-connected, locally path-connected space such that its fundamental group $\pi_1(X)$ is finite. Let $f : X \to S^1$ be a continuous map.

a. Let $x_0, x_1 \in X$ and $u, v : I \to X$ be two paths from x_0 to x_1. Prove that the paths $f \circ u$ and $f \circ v$ in S^1 are homotopic.

b. Let $p : \mathbb{R} \to S^1$ be the standard covering $t \mapsto e^{2\pi i t}$. Prove that f can be lifted to a continuous map $\tilde{f} : X \to \mathbb{R}$ such that $p \circ \tilde{f} = f$.

c. Prove that f is homotopic to a constant map.
Hint: Don't forget to explain where the fact that X is locally path-connected is used.

6. Let $f : S^1 \to S^1$ be a continuous map and X be its mapping cylinder. Show that X has a structure of a cellular space and give its cellular decomposition explicitly. What is the minimal number of cells that one needs for this?

7. Let X be a space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other one.

a. Find a cellular decomposition of X (Hint: verify that the Euler characteristic is what it is supposed to be);

b. Compute the fundamental group of X.