Math 76 Project 2

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June 24, 2017

1 Introduction

This project is an extension of the REU project at Michigan State University from last year: https://sites.google.com/site/suriem2016mathmri/home. It also uses the results from the following papers:

1. T. Sanders, A. Gelb and R. Platte, *Composite SAR Imaging Using Sequential Joint Sparsity*, Journal of Computational Physics, **338** 357–370 (2017) DOI: 10.1016/j.jcp.2017.02.071.
2. D. Denker and A. Gelb, *Edge Detection of Piecewise Smooth Functions from Under-Sampled Fourier Data Using Variance Signatures*, SIAM Journal on Scientific Computing, **39:2** 559-592 (2017).
3. W. Stefan, A. Viswanathan, A. Gelb and R. Renaut, *Sparsity Enforcing Edge Detection Method for Blurred and Noisy Fourier Data*, Journal of Scientific Computing, **50:3** 536-556 (2012).
4. A. Viswanathan, D. Cochran and A. Gelb, *Iterative Design of Concentration Factors for Edge Detection*, Journal of Scientific Computing, **51:3** 631-649 (2012).
5. A. Gelb and G. Song, *Detecting Edges from Non-uniform Fourier Data Using Fourier Frames,* Journal of Scientific Computing, **71:2** 737–758 (2017) DOI:10.1007/s10915-016-0320-8.

2 Problem set up

Let $f(x)$ be a periodic piecewise smooth function (signal) on an interval $[-1,1]$. Assume we have M vectors $\hat{\mathbf{f}}_1,\cdots,\hat{\mathbf{f}}_M$. Each $\hat{\mathbf{f}}_m$ contains the first $2N+1$ Fourier coefficients of f, that is $\hat{f}_k = \frac{1}{2}\int_{-1}^{1} f(x)exp(-ik\pi x)dx$, $k=-N,\cdots,N$. (You can do a linear transformation if you prefer to use a different interval $[a,b]$.) There are several applications for which Fourier data are collected, such as MRI, ultrasound, and synthetic aperture radar (SAR). One of the main problems is that N, relating to the number of Fourier samples collected, may not be sufficient to recover information needed from the underlying signal. We ask the following question:

- Is it possible to extract more Fourier data (that is outside the interval $[-N,N]$) by processing the existing given data?

This is called the *super-resolution* problem. The student participants in the 2016 MSU REU program considered this problem for $M = 1$. We will now consider $M > 1$. Their approach was to use the relationship between the *edges* of f (see **Project 1** for definition of edges) and the *high frequency* Fourier coefficients, $|k| > N$. Of course they had to first approximate the edges of f from the given Fourier data, $\hat{f}_k, k = -N, \cdots, N$. The process is described on the MSU REU website and the method to determine edges from Fourier data can be found in (Viswanathan, Cochran, and Gelb) and references within that paper.

Once we have recovered more Fourier data, we can hopefully also recover a better underlying signal. We will break this up into different tasks. Everyone should be able to implement the method provided on the MSU REU website, but the other tasks in **Part 1** do not have to all be rigorously analyzed (although you should have a general idea of how well the method works in different scenarios). Some of you may choose to more look more closely into the $M = 1$ case.

I encourage you to consult each other when working on this project, especially in terms of programming. Remember, everyone is coming into this with different backgrounds, so you will learn a lot from talking together. Don't hesitate to use canned packages, no one should reinvent the wheel! Also, more than one person can work on the same question, there are many ways to go about investigating the solution.

- **Part 1:** First assume that $M = 1$. Consult the MSU REU website listed above to recover $\hat{f}_k$ for $|k| > N$. You should ask yourself the following questions:
  - **Method utility:** How many coefficients can you recover? That is what is the accuracy of recovering the $N + n$th coefficient as n increases? Also, there are several ways that you can determine the jump function $[f]$ from the Fourier data, which will feed back into the algorithm. Which method seems to work the best? Does it matter that much?
  - **Effects of noise:** Suppose the given data are instead $\hat{g}_k = \hat{f}_k + \epsilon_k$, $k = -N, \cdots, N$ where ϵ_k is complex i.i.d. Gaussian noise. How does that affect your results?
  - **Missing data bands:** Suppose bands of data, $[N_1, N_2]$ (and $[-N_2, N_1]$ where $N > N_2 > N_1 > 0$, are missing. Will the method still work?
  - **Function reconstruction:** Provide examples where super resolution plays a critical role in being able to reconstruct the underlying signal. Note that due to the piecewise smooth nature of the underlying signal you will not achieve high accuracy, but you should still be able to see the effects (you will have to come to discuss this with me in my office, the explanation is too long for this write up).
  - **Two dimensions:** Can you think of a better way to approach this problem in $2D$?
- **Part 2:** Now assume that $M > 1$. Consult **Project 1** to see how multiple measurement vectors might improve the results. We would like to be able to answer the same questions as in the $M = 1$ case. But now we can add to that
  - Is there an optimal value for M? Since the procedure is more costly, how many vectors do I need?
  - A related question: What is the relationship between M and N?
  - Another related question: What is the relationship between M and noise, blur, missing data?
- **Part 3:** Instead of the variance method described in **Project 1**, one can also use the joint sparsity approach (see Sanders, Gelb and Platte and references therein). Do a comparison to see which one might be more effective.

- **Part 4:** Applications: We can again consider the cocktail party. Does super-resolution help? Or you can come up with another application.
- **Part 5:** There may be some applications where you have different modalities of data collection that may make it easier to determine the edges of f. Using the ideas from **Project 1** or from joint sparsity, can you recover high frequency Fourier data?