Dartmouth College
Mathematics 81

The following exercises are to remind you of (or sharpen your skills regarding) material from Math 71. First, I give a few results which you may use without proof, but if they are unfamiliar, you should read the relevant material in your text.

Let $f \in \mathbb{Z}[x]$. f is called *primitive* if and only if the gcd of its coefficients is 1. For example $2x^2 - 6x + 3$ is primitive in $\mathbb{Z}[x]$. The following two theorems are essentially (if not in fact) equivalent to Gauss's lemma over $\mathbb{Q}$:

Theorem: Let $f \in \mathbb{Z}[x]$. Then f is irreducible in $\mathbb{Z}[x]$ if and only if f is primitive in $\mathbb{Z}[x]$ and irreducible in $\mathbb{Q}[x]$.

Theorem: Let $f \in \mathbb{Z}[x]$, and suppose that $f = gh$ for two polynomials $g, h \in \mathbb{Q}[x]$. Then $f = g_0 h_0$ for polynomials $g_0, h_0 \in \mathbb{Z}[x]$ with $\deg(g) = \deg(g_0)$ and $\deg(h) = \deg(h_0)$. In particular g_0 and h_0 are integer scalar multiples of g and h respectively.

Here are the problems to work on:

1. The following exercise is meant to refresh your memory about ideals and quotient rings. For each of the ideals I listed below, determined whether the ring $\mathbb{Z}[x]/I$ has zero divisors, is an integral domain, or is a field (and hence whether the ideal I is not prime, prime, or maximal). If the quotient is not an integral domain, find zero divisors. If the quotient is not a field, then I is not maximal, so find a maximal ideal M with $I \subsetneq M$, and justify that M is maximal.

   Hint: You may use without proof the very handy fact that if $f \in \mathbb{Z}[x]$ and $m \in \mathbb{Z}$, then $\mathbb{Z}[x]/(m, f(x)) \cong (\mathbb{Z}/m\mathbb{Z})[x]/(\bar{f}(x))$, where $\bar{f}(x)$ is the polynomial in $(\mathbb{Z}/m\mathbb{Z})[x]$ obtained from f by reducing the coefficients modulo m.

   (a) $I = (x^3 + 2)$
   (b) $I = (5, x^3 + 2)$
   (c) $I = (7, x^3 + 2)$

2. Let $\mathbb{F}_{11} = \mathbb{Z}_{11} = \mathbb{Z}/11\mathbb{Z}$ be a (actually the) field with 11 elements, and set $K = \mathbb{F}_{11}[x]/(x^2 + 1)$ and $L = \mathbb{F}_{11}[y]/(y^2 + 2y + 2)$.

   (a) Show that K and L are both fields with 121 elements.
   (b) For $p(x) \in \mathbb{F}_{11}[x]$, let $\overline{p(x)}$ denote its image in K. Show that the map from $K \to L$ which takes $\overline{p(x)} \mapsto \overline{p(y+1)}$ is well-defined and a ring homomorphism. Finally show that the map is an isomorphism.