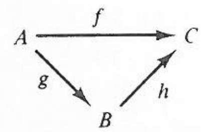
4. Prove the following, in which all structures are assumed to be L-structures for some fixed signature L.

(a) Every homomorphism $f\colon A \to C$ can be factored as $f = hg$ for some surjective homomorphism $g\colon A \to B$ and extension $h\colon B \to C$:

(2.6)



The unique structure B is called the **image** *of g,* img *for short. More generally we say that an L-structure B is a* **homomorphic image** *of A if there exists a surjective homomorphism $g\colon A \to B$.*

(b) Every embedding $f\colon A \to C$ can be factored as $f = hg$ where g is an extension and h is an isomorphism. *This is a small piece of set theory which often gets swept under the carpet. It says that when we have embedded A into C, we can assume that A is a substructure of C. The embedding of the rationals in the reals is a familiar example.*

5. Let A and B be L-structures with A a substructure of B. A **retraction** from B to A is a homomorphism $f\colon B \to A$ such that $f(a) = a$ for every element a of A. Show (a) if $f\colon B \to A$ is a retraction, then $f^2 = f$, (b) if B is any L-structure and f an endomorphism of B such that $f^2 = f$, then f is a retraction from B to a substructure A of B.

6. Let L be a finite signature with no function symbols. (a) Show that every finitely generated L-structure is finite. (b) Show that for each $n < \omega$ there are up to isomorphism only finitely many L-structures of cardinality n.

7. Show that we can give fields a signature which makes substructure = subfield and homomorphism = field embedding, provided we are willing to let 0^{-1} be 0. *Most mathematicians seem to be unwilling, and so the normal custom is to give fields the same signature as rings.*

8. Many model theorists require the domain of any structure to be non-empty. How would this affect Lemma 1.2.2 and the definition of $\langle Y \rangle_B$?

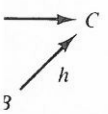
9. Give an example of a structure of cardinality ω_2 which has a substructure of cardinality ω but no substructure of cardinality ω_1.

1.3 Terms and atomic formulas

In Chapter 2 we shall introduce a formal language for talking about L-structures. This language will be built up from the atomic formulas of L, which we must now define.

Each atomic formula will be a string of symbols including symbols of L. Since the symbols in L can be any kinds of object and not necessarily written res are assumed to be L-structures for some

n be factored as $f = hg$ for some surjective
$: B \to C$:



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atomic formulas

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expressions, the idea of a 'string of symbols' has to be taken with a pinch of set-theoretic coding.

Terms

Every language has a stock of **variables**. These are symbols written v, x, y, z, t, x_0, x_1 etc., and one of their purposes is to serve as temporary labels for elements of a structure. Any symbol can be used as a variable, provided it is not already being used for something else. The choice of variables is never important, and for theoretical purposes many model theorists restrict them to be the expressions v_0, v_1, v_2, . . . with natural number or ordinal subscripts.

The **terms** of the signature L are strings of symbols defined as follows (where the symbols '(', ')' and ',' are assumed not to occur anywhere in L – henceforth points like this go without saying).

(3.1) Every variable is a term of L.

(3.2) Every constant of L is a term of L.

(3.3) If $n > 0$, F is an n-ary function symbol of L and $t_1, \ldots, t_n$ are terms of L then the expression $F(t_1, \ldots, t_n)$ is a term of L.

(3.4) Nothing else is a term of L.

A term is said to be **closed** (computer scientists say **ground**) if no variables occur in it. The **complexity** of a term is the number of symbols occurring in it, counting each occurrence separately. (The important point is that if t occurs as part of s then s has higher complexity than t.)

If we introduce a term t as $t(\bar{x})$, this will always mean that $\bar{x}$ is a sequence $(x_0, x_1, \ldots)$, possibly infinite, of distinct variables, and every variable which occurs in t is among the variables in $\bar{x}$. Later in the same context we may write $t(\bar{s})$, where $\bar{s}$ is a sequence of terms $(s_0, s_1, \ldots)$; then $t(\bar{s})$ means the term got from t by putting s_0 in place of x_0, s_1 in place of x_1, etc., throughout t. (For example if $t(x, y)$ is the term $y + x$, then $t(0, 2y)$ is the term $2y + 0$ and $t(t(x, y), y)$ is the term $y + (y + x)$.)

To make variables and terms stand for elements of a structure, we use the following convention. Let $t(\bar{x})$ be a term of L, where $\bar{x} = (x_0, x_1, \ldots)$. Let A be an L-structure and $\bar{a} = (a_0, a_1, \ldots)$ a sequence of elements of A; we assume that $\bar{a}$ is at least as long as $\bar{x}$. Then $t^A(\bar{a})$ (or $t^A[\bar{a}]$ when we need a more distinctive notation) is defined to be the element of A which is named by t when x_0 is interpreted as a name of a_0, and x_1 as a name of a_1, and so on. More precisely, using induction on the complexity of t,

(3.5) if t is the variable x_i then $t^A[\bar{a}]$ is a_i,

(3.6) if t is a constant c then $t^A[\bar{a}]$ is the element c^A, (3.7) if t is of the form $F(s_1, \ldots, s_n)$ where each s_i is a term $s_i(\bar{x})$, then $t^A[\bar{a}]$ is the element $F^A(s_1{}^A[\bar{a}], \ldots, s_n{}^A[\bar{a}])$.

(Cf. (1.2), (1.4) above.) If t is a closed term then $\bar{a}$ plays no role and we simply write t^A for $t^A[\bar{a}]$.

Atomic formulas

The **atomic formulas** of L are the strings of symbols given by (3.8) and (3.9) below.

(3.8) If s and t are terms of L then the string $s = t$ is an atomic formula of L.

(3.9) If $n > 0$, R is an n-ary relation symbol of L and $t_1, \ldots, t_n$ are terms of L then the expression $R(t_1, \ldots, t_n)$ is an atomic formula of L.

(Note that the symbol '=' is not assumed to be a relation symbol in the signature.) An **atomic sentence** of L is an atomic formula in which there are no variables.

Just as with terms, if we introduce an atomic formula ϕ as $\phi(\bar{x})$, then $\phi(\bar{s})$ means the atomic formula got from ϕ by putting terms from the sequence $\bar{s}$ in place of all occurrences of the corresponding variables from $\bar{x}$.

If the variables $\bar{x}$ in an atomic formula $\phi(\bar{x})$ are interpreted as names of elements $\bar{a}$ in a structure A, then ϕ makes a statement about A. The statement may be true or false. If it's true, we say ϕ is **true of $\bar{a}$ in A**, or that **$\bar{a}$ satisfies ϕ in A**, in symbols

$$A \models \phi[\bar{a}] \quad \text{or equivalently} \quad A \models \phi(\bar{a}).$$

We can give a formal definition of this relation $\models$. Let $\phi(\bar{x})$ be an atomic formula of L with $\bar{x} = (x_0, x_1, \ldots)$. Let A be an L-structure and $\bar{a}$ a sequence $(a_0, a_1, \ldots)$ of elements of A; we assume that $\bar{a}$ is at least as long as $\bar{x}$. Then

(3.10) if ϕ is the formula $s = t$ where $s(\bar{x})$, $t(\bar{x})$ are terms, then $A \models \phi[\bar{a}]$ iff $s^A[\bar{a}] = t^A[\bar{a}]$,

(3.11) if ϕ is the formula $R(s_1, \ldots, s_n)$ where $s_1(\bar{x}), \ldots, s_n(\bar{x})$ are terms, then $A \models \phi[\bar{a}]$ iff the ordered n-tuple $(s_1{}^A[\bar{a}], \ldots, s_n{}^A[\bar{a}])$ is in R^A.

(Cf. (1.3) above.) When ϕ is an atomic sentence, we can omit the sequence $\bar{a}$ and write simply $A \models \phi$ in place of $A \models \phi[\bar{a}]$.

We say that A is a **model** of ϕ, or that ϕ is **true in** A, if $A \models \phi$. When T is a set of atomic sentences, we say that A is a **model** of T (in symbols, $A \models T$) if A is a model of every atomic sentence in T.

Theorem 1.3.1. *Let A and B be L-structures and f a map from* $\mathrm{dom}(A)$ *to* $\mathrm{dom}(B)$.

(a) *If f is a homomorphism then, for every term $t(\bar{x})$ of L and tuple $\bar{a}$ from A, $f(t^A[\bar{a}]) = t^B[f\bar{a}]$.*

(b) *f is a homomorphism if and only if, for every atomic formula $\phi(\bar{x})$ of L and tuple $\bar{a}$ from A,*

(3.12) $$A \models \phi[\bar{a}] \Rightarrow B \models \phi[f\bar{a}].$$

(c) *f is an embedding if and only if, for every atomic formula $\phi(\bar{x})$ of L and tuple $\bar{a}$ from A,*

(3.13) $$A \models \phi[\bar{a}] \Leftrightarrow B \models \phi[f\bar{a}].$$

Proof. (a) is easily proved by induction on the complexity of t, using (3.5)–(3.7).

(b) Suppose first that f is a homomorphism. As a typical example, suppose $\phi(\bar{x})$ is $R(s, t)$, where $s(\bar{x})$ and $t(\bar{x})$ are terms. Assume $A \models \phi[\bar{a}]$. Then by (3.11) we have

(3.14) $$(s^A[\bar{a}],\ t^A[\bar{a}]) \in R^A.$$

Then by part (a) and the fact that f is a homomorphism (see (2.2) above),

(3.15) $$(s^B[f\bar{a}],\ t^B[f\bar{a}]) = (f(s^A[\bar{a}]),\ f(t^A[\bar{a}])) \in R^B.$$

Hence $B \models \phi[f\bar{a}]$ by (3.11) again. Essentially the same proof works for every atomic formula ϕ.

For the converse, again we take a typical example. Assume that (3.12) holds for all atomic ϕ and sequences $\bar{a}$. Suppose $(a_0, a_1) \in R^A$. Then writing $\bar{a}$ for (a_0, a_1), we have $A \models R(x_0, x_1)[\bar{a}]$. Then (3.12) implies $B \models R(x_0, x_1)[f\bar{a}]$, which by (3.11) implies that $(fa_0, fa_1) \in R^B$ as required. Thus f is a homomorphism.

(c) is proved like (b), but using (2.4) in place of (2.2). □

A variant of Theorem 1.3.1(c) is often useful. By a **negated atomic formula** of L we mean a string $\neg\phi$ where ϕ is an atomic formula of L. We read the symbol $\neg$ as 'not' and we define

(3.16) $$`A \models \neg\phi[\bar{a}]\text{' holds} \quad \text{iff} \quad `A \models \phi[\bar{a}]\text{' doesn't hold.}$$

where A is any L-structure, ϕ is an atomic formula and $\bar{a}$ is a sequence from A. A **literal** is an atomic or negated atomic formula; it's a **closed literal** if it contains no variables.

Corollary 1.3.2. *Let A and B be L-structures and f a map from* $\mathrm{dom}(A)$ *to* $\mathrm{dom}(B)$. *Then f is an embedding if and only if, for every literal $\phi(\bar{x})$ of L and sequence $\bar{a}$ from A,*

$$A \vDash \phi[\bar{a}] \Rightarrow B \vDash \phi[f\bar{a}]. \tag{3.17}$$

Proof. Immediate from (c) of the theorem and (3.16). □

The term algebra

The Delphic oracle said 'Know thyself'. Terms can do this – they can describe themselves. Let L be any signature and X a set of variables. We define the **term algebra of L with basis X** to be the following L-structure A. The domain of A is the set of all terms of L whose variables are in X. We put

(3.18) $c^A = c$ for each constant c of L,

(3.19) $F^A(\bar{t}) = F(\bar{t})$ for each n-ary function symbol F of L and n-tuple $\bar{t}$ of elements of $\mathrm{dom}(A)$,

(3.20) R^A is empty for each relation symbol R of L.

The term algebra of L with basis X is also known as the **absolutely free L-structure with basis X**.

Exercises for section 1.3

1. Let B be an L-structure and Y a set of elements of B. Show that $\langle Y \rangle_B$ consists of those elements of B which have the form $t^B[\bar{b}]$ for some term $t(\bar{x})$ of L and some tuple $\bar{b}$ of elements of Y. [Use the construction of $\langle Y \rangle_B$ in the proof of Theorem 1.2.3.]

2. (a) If $t(x, y, z)$ is $F(G(x, z), x)$, what are $t(z, y, x)$, $t(x, z, z)$, $t(F(x, x), G(x, x), x)$, $t(t(a, b, c), b, c)$?
(b) Let A be the structure $\langle \mathbb{N}, +, \cdot, 0, 1 \rangle$ where $\mathbb{N}$ is the set of natural numbers 0, 1, ... Let $\phi(x, y, z, u)$ be the atomic formula $x + z = y \cdot u$ and let $t(x, y)$ be the term $y \cdot y$. Which of the following are true? $A \vDash \phi[0, 1, 2, 3]$; $A \vDash \phi[1, 5, t^A[4, 2], 1]$; $A \vDash \phi[9, 1, 16, 25]$; $A \vDash \phi[56, t^A[9, t^A[0, 3]], t^A[5, 7], 1]$. [Answers: $F(G(z, x), z)$, $F(G(x, z), x)$, $F(G(F(x, x), x), F(x, x))$, $F(G(F(G(a, c), a), c), F(G(a, c), a))$; no, yes, yes, no.]

3. Let A be an L-structure, $\bar{a} = (a_0, a_1, \ldots)$ a sequence of elements of A and $\bar{s} = (s_0, s_1, \ldots)$ a sequence of closed terms of L such that, for each i, $s_i^A = a_i$. Show (a) for each term $t(\bar{x})$ of L, $t(\bar{s})^A = t^A[\bar{a}]$, (b) for each atomic formula $\phi(\bar{x})$ of L, $A \vDash \phi(\bar{s}) \Leftrightarrow A \vDash \phi[\bar{a}]$.