Week 1 Friday

Set Theory

Due Friday Jan 15th

This homework will explore the central mathematical step in the Burali-Forti paradox: if the aggregate of all ordinals were a set, then it would itself be an ordinal. First, let's remind ourselves of some definitions.

Definition. Let A be a set. The binary relation $< \subseteq A \times A$ is said to be a strict linear order if $<$ satisfies the following:

$$\text{(i) (totality) } \forall a, b \in A \ (a \neq b \rightarrow (a < b \vee b < a))$$

$$\text{(ii) (antisymmetry) } \forall a, b \in A \ \neg(a < b \wedge b < a)$$

$$\text{(iii) (transitivity) } \forall a, b, c \in A \ ((a < b \wedge b < c) \rightarrow a < c)$$

Definition. Let A be a set, let $<$ be a strict linear order on A, and let $S \subseteq A$. An element $a_0 \in S$ is $<$-least in S if $\forall b \in S \ (a_0 = b \vee a_0 < b)$.

Definition. Let A be a set. A binary relation $<$ is said to be a well-ordering if it satisfies (i), (ii), and (iii) above and if every non-empty subset of A has a $<$-least element. That is:

$$\text{(iv) } \forall S((S \subseteq A \wedge S \neq \emptyset) \rightarrow \exists a_0 \in S \ \forall b \in S(a_0 = b \vee a_0 < b))$$

A well-order seems like a swell order, but if it could be expressed in the simplest way possible that would be even neater.

Definition. A set α is called an ordinal if:

$$\text{(i) the element-of relation } \in \text{ is a well-order on } \alpha$$

$$\text{(ii) } \forall \beta \in \alpha \ (\gamma \in \beta \rightarrow \gamma \in \alpha)$$

Definition. *Let α be an ordinal. The successor of α, denoted $\alpha + 1$, is the set $\alpha \cup \{\alpha\}$.*

If you check the logic carefully, $\emptyset$ is an ordinal for trivial reasons. As suggested in class, it represents 0. Ordinals start at nothing, and ordinals can always keep going.

Facts. *(i) Let α and β be distinct ordinals. Either $\alpha \in \beta$ or $\beta \in \alpha$ but not both.*

(ii) If α is an ordinal, then $\alpha + 1$ is also an ordinal.

(iii) If α is an ordinal and $\beta \in \alpha$, then β is an ordinal.

***1)** Prove the following using the facts above:

Claim. *Suppose $\mathcal{O} = \{ \alpha \mid \alpha \text{ is an ordinal} \}$ is a set. Then $\mathcal{O}$ is an ordinal.*

(Hint: Let $A \subseteq \mathcal{O}$ be non-empty. So $\exists \alpha \in A$. Either α is $\in$-least in A or ...)