# Week 1 Wednesday

Set Theory

Due Friday Jan 15th

Using indexed families of sets for generalized arithmetic serves mostly to give structure to generalized products. However, generalized unions and intersections do not always benefit from this extra structure. So we make more practical definitions with a sleeker notation.

**Definition.** *Let $\mathcal{F}$ be a collection of sets. The union of $\mathcal{F}$, denoted $\bigcup \mathcal{F}$, is the set of all elements of sets in $\mathcal{F}$. That is to say,*

$$\bigcup \mathcal{F} = \{\ x \mid \exists S(x \in S \wedge S \in \mathcal{F})\ \}$$

*Similarly, the intersection of $\mathcal{F}$ is the set of all "things" that are elements of all the sets in $\mathcal{F}$. That is to say,*

$$\bigcap \mathcal{F} = \{\ x \mid \forall S(S \in \mathcal{F} \to x \in S)\ \}$$

**Instructions:** Note that the truth of each set equality below hinges essentially on the logical equivalence of two logical formulae. So, choose one of the set equalities below to prove in reasonably full detail. Then for each of the other set equalities, you need only give the corresponding logical formulae (as formal as you can write them) whose equivalence demonstrates that set equality.

**1) Generalized distributive laws:**

Let $S$ be a set and let $\mathcal{F}$ be a collection of sets.

$$\text{(i) } S \cap \bigcup \mathcal{F} = \bigcup_{X \in \mathcal{F}} S \cap X$$

$$\text{(ii) } S \cup \bigcap \mathcal{F} = \bigcap_{X \in \mathcal{F}} S \cup X$$

**2) Generalized De Morgan's laws:**

Let $S$ be a set and let $\mathcal{F}$ be a collection of sets.

$$\text{(i) } S \setminus \bigcup \mathcal{F} = \bigcap_{X \in \mathcal{F}} S \setminus X$$

$$\text{(ii) } S \setminus \bigcap \mathcal{F} = \bigcup_{X \in \mathcal{F}} S \setminus X$$

**Parting Words:** We can think of a set $A$ as describing a property that can be ascribed to objects. In particular, for each $x$ either $x \in A$ or $x \notin A$ and not both. Similarly, if $P$ is a well-defined property then for each $x$ the statement $P(x)$ is true or false and not both. Considering this parallel, we see that the set equalities and the corresponding logical equivalences are, themselves, equivalent.