Set Theory week 4 Due Monday Feb. 8th

Friday: Small Countable Ordinals

1) For the following formulas, construct the smallest α that satisfies the formula. Prove its minimality.

(i) $1+\alpha=\alpha$

(ii) $\omega+\alpha=\alpha$

(iii) $\omega\cdot\alpha=\alpha$, $\alpha\neq 0$

(iv) $\omega^{\alpha}=\alpha$

Definition: We can formally define an operation of ordinal tetration that captures transfinite iterations of exponentiation.

For $\alpha>0$

(init.) $\alpha\cdot_4 1=\alpha$

(succ.) $\alpha\cdot_4(\beta+1)=(\alpha\cdot_4\beta)^{\alpha}$

(limit) $\alpha\cdot_4\gamma=\bigcup_{\delta<\gamma}\alpha\cdot_4\delta$

* **2)** Define by transfinite recursion a sequence ε_α:

(init) $\varepsilon_0=\omega$

(succ.) $\varepsilon_{\beta+1}=$ the least α such that $\alpha=\varepsilon_\beta^{\alpha}$

(limit) $\varepsilon_\gamma=\bigcup_{\delta<\gamma}\varepsilon_\delta$

Give an explicit expression of ε_α in terms of α and known ordinal operations. Prove that your formula satisfies the recursive definition. (Thus, the sequence exists as $\{\alpha \,/\, \alpha=\varepsilon_\beta^{\alpha}\}\neq\emptyset$)

* **3)** Describe the smallest ordinal α such that $\alpha=\omega\cdot_4\alpha$.

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4) Define, for $n \geq 4$, an operation $\bullet_{n+1}$ which formalizes transfinite iteration of the operation $\bullet_n$.

For each $n \geq 4$, describe how to obtain the smallest α such that $\alpha = \omega \bullet_n \alpha$.