Math 8, Fall 2019, Exam I Practice Solutions

1. Use the definition of limit to show that $\lim_{n\to\infty} \frac{n^2+1}{n^2} = 1.$

Solution: Given $\varepsilon > 0$, we must find an N such that

$$n > N \implies \left|1 - \frac{n^2+1}{n^2}\right| < \varepsilon.$$

Doing some algebraic manipulations, we see that

$$\left|1 - \frac{n^2+1}{n^2}\right| < \varepsilon \iff \left|\frac{-1}{n^2}\right| < \varepsilon \iff \frac{1}{n^2} < \varepsilon \iff \frac{1}{\varepsilon} < n^2 \iff \frac{1}{\sqrt{\varepsilon}} < |n|.$$

Choose $N > \frac{1}{\sqrt{\varepsilon}}$. Then

$$n > N \implies n > \frac{1}{\sqrt{\varepsilon}} \implies |n| > \frac{1}{\sqrt{\varepsilon}} \implies \left|1 - \frac{n^2+1}{n^2}\right| < \varepsilon$$

(the last step is by our earlier algebraic manipulations). This is what we needed.

2. (a) Find the sum: $\sum_{k=4}^{\infty} \frac{2\cdot 3^k}{5^{2k+1}} =$

Solution: This is a geometric series, with first term $a = \frac{2\cdot 3^4}{5^9}$ and ratio of successive terms $r = \frac{3}{25}$. Since $|r| < 1$, the series converges, and its sum is

$$\frac{a}{1-r} = \frac{\frac{2\cdot 3^4}{5^9}}{1-\frac{3}{25}} = \frac{\frac{2\cdot 3^4}{5^9}}{\frac{22}{25}} = \frac{3^4}{11\cdot 5^7}.$$

(b) Determine whether the series converges: $\sum_{k=1}^{\infty} \frac{k^3 2^k}{3^{k-1}}$

Solution: We use the ratio test.

$$\lim_{n\to\infty} \left| \frac{\frac{(k+1)^3 2^{(k+1)}}{3^k}}{\frac{k^3 2^k}{3^{k-1}}} \right| = \frac{2}{3} \lim_{k\to\infty} \left|\frac{k+1}{k}\right|^3 = \frac{2}{3} < 1$$

so the series converges.

(c) Find the limit: $\lim_{n\to\infty} \frac{\ln(n)}{n} =$

Solution: Using l'Hopital's rule, we see

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0,$$

and therefore $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$.

3. (a) Compute $T_{10}(x)$, the 10th degree Taylor Polynomial, of $\sin(x)\cos(x)$ centered at $x = 0$.

Solution: First we compute enough derivatives of $f(x) = \sin(x)\cos(x)$ to see the pattern:
$f'(x) = \cos^2 x - \sin^2 x$; $f''(x) = -4\cos x \sin x$.
Since we are back to a multiple of f, we can already find the pattern:
if k is even $f^{(k)}(x) = (-1)^{\frac{k}{2}} 2^k \sin(x)\cos(x)$;
if k is odd $f^{(k)}(x) = (-1)^{\frac{k-1}{2}} 2^{k-1}(\cos^2(x) - \sin^2(x))$.
Evaluating at 0, if k is even $f^{(k)}(0) = 0$, and if k is odd $f^{(k)}(0) = (-1)^{\frac{k-1}{2}} 2^{k-1}$.
Using the formula for Taylor polynomials, this gives us

$$T_{10}(x) = \frac{x}{1} - \frac{4x^3}{3!} + \frac{16x^5}{5!} - \frac{2^6 x^7}{7!} + \frac{2^8 x^9}{9!}.$$

(b) Find the radius of convergence of the corresponding Taylor series using Taylor's Inequality.

Taylor's Inequality: If $T_n(x)$ is the degree n Taylor polynomial of f centered at $x = a$, and $B_{n+1} \geq |f^{(n+1)}(w)|$ for all w between a and x, then $|f(x) - T_n(x)|$ is at most $\frac{B_{n+1}}{(n+1)!}|x-a|^{n+1}$.

Solution: If k is even, we have $f^{(k)}(x) = (-1)^{\frac{k}{2}} 2^k \sin(x)\cos(x)$, and since $|\sin(x)|$ and $|\cos(x)|$ are at most 1, the largest $|f^{(k)}(x)|$ could be, for any x, is 2^k.
If k is odd, we have $f^{(k)}(x) = (-1)^{\frac{k-1}{2}} 2^{k-1}(\cos^2(x) - \sin^2(x))$, and since $|\sin(x)|$ and $|\cos(x)|$ are at most 1, the largest $|\cos^2(x) - \sin^2(x)|$ could be is 2, and again the largest $|f^{(k)}(x)|$ could be, for any x, is 2^k.
Therefore, for any x, we can take $B_{n+1} = 2^{n+1}$.

To find the radius of convergence for the Taylor series, we need to find for which x the error approaches 0 as $n \to \infty$. Using Taylor's error formula, we are looking at

$$\lim_{n\to\infty} \frac{B_{n+1}}{(n+1)!}|x-a|^{n+1} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!}|x|^{n+1} = \lim_{n\to\infty} \frac{|2x|^{n+1}}{(n+1)!}.$$

Since $|2x|$ is a constant (for any particular x), and we have seen that if c is a constant then $\lim_{n\to\infty} \frac{c^n}{n!} = 0$, we know the error approaches 0 for every x. Therefore the Taylor series converges for every x, and the radius of convergence is ∞.

4. Between noon and 2 PM a car is traveling with velocity at t hours past noon equal to $65 - 2t$ miles per hour.

   (a) Over a very short time interval of length Δt hours, at approximately t hours past noon, approximately how far does the car travel?

   Solution: Near time t the car is traveling at approximately $65 - 2t$ miles per hour. It travels for Δt hours so it travels approximately $(65 - 2t)\,\Delta t$ miles.

   (b) Write down a Riemann sum representing the total distance the car travels between noon and 2 PM.

   (Don't forget to explain what you are doing. In particular, if your formula includes things like "Δt" or "x_i", you should say what they mean.)

   Solution: Divide the 2 hour interval between noon and 2 PM into n-many short intervals of length Δt. For each i between 1 and n choose a time t_i in the i^{th} time interval. By part (a), during the i^{th} time interval the car travels approximately $(65 - 2t_i)\,\Delta t$ miles. Therefore the total distance it travels is approximately

$$\sum_{i=1}^{n} (65 - 2t_i)\,\Delta t \text{ miles.}$$

5. (a) A bucket of sand with total mass 2 kg is lifted by a lightweight rope at a constant speed of 10 km/h (kilometers per hour) to the height of 16 m. What is the total work done, in joules (J)?

(Near the earth's surface, the force of gravity acting on an object of mass m kg is gm newtons, where $g \approx 9.8$.)

Solution: The force on the bucket is its mass times g, or $2g$ newtons. It is lifted a distance of 16 meters, so we compute the work by multiplying force times distance,

$$32g \text{ J } \approx 32(9.8) \text{ J.}$$

(Note that the speed at which the bucket is lifted is irrelevant.)

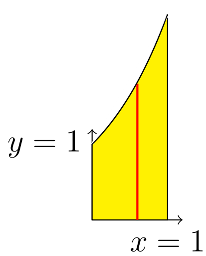
(b) If the sand is leaking out, so that when the bucket reaches height h meters, its mass is $(2 - .01h)$ kilograms, what is the total work done?

Solution: The bucket is lifted from height $h = 0$ meters to height $h = 16$ meters (measuring height from its starting point), and at height h the mass of the bucket is $(2 - .01h)$ kilograms, so the force on the bucket is $(2 - .01h)g$ newtons. Since force is variable, and we can write force as a function of distance (h), we can compute work by integrating force with respect to distance:

$$\int_0^{16} (2 - .01h)g\, dh \text{ J } = g(2h - .005h^2)\Big|_{h=0}^{h=16} \text{ J} = g(32 - .005(16)^2) \text{ J.}$$

6. A solid is generated by revolving around the y-axis the region bounded by $x = 1$, $y = e^x$, the y-axis and the x-axis. Compute its volume.

Solution: The region we revolve around the y-axis is pictured here:



It turns out to work out better to use volumes by shells than volumes by slicing for this region. As the red vertical line at point x on the x-axis is revolved around the y-axis, it sweeps out a cylinder of radius x and height the length of the red line, or e^x. Therefore this cylinder has area $2\pi x e^x$. We compute the volume by integrating this area,

$$\int_0^1 2\pi x e^x\, dx.$$

We can integrate xe^x by parts, using $u = x$, $dv = e^x dx$, so $du = dx$ and $v = e^x$:

$$\int x\, e^x\, dx = uv - \int v\, du = xe^x - \int e^x\, dx = xe^x - e^x + C.$$

Therefore our volume is

$$\int_0^1 2\pi x e^x\, dx = 2\pi(xe^x - e^x)\Big|_{x=0}^{x=1} = 2\pi.$$

7. Compute $\int x^3 \cos(x) dx$.

   Solution: This problem simply requires several applications of integration by parts. First, by setting $u_1 = x^3$ and $dv_1 = \cos(x)dx$, we get $du_1 = 3x^2 dx$ and $v_1 = \sin(x)$ so

$$\int x^3 \cos(x) dx = x^3 \sin(x) - 3 \int x^2 \sin(x) dx$$

   Now we must solve this second integral: $I_2 = \displaystyle\int x^2 \sin(x) dx$. Like before, set $u_2 = x^2$ and $dv_2 = \sin(x)dx$, so that $du_2 = 2xdx$ and $v_2 = -cos(x)$. Then

$$I_2 = \int x^2 \sin(x) dx = -x^2 \cos(x) + 2 \int x \cos(x) dx$$

   We have one final integral to solve: $I_3 = \displaystyle\int x \cos(x) dx$. Set $u_3 = x$ and $dv_3 = \cos(x)dx$. Then $du_3 = dx$ and $v_3 = \sin(x)$, so

$$I_3 = \int x \cos(x) dx = x \sin(x) - \int \sin(x) dx = x \sin(x) + \cos(x) + C$$

   Now we put all the pieces together:

$$
\begin{aligned}
\int x^3 \cos(x) dx &= x^3 \sin(x) - 3 I_2 \\
&= x^3 \sin(x) - 3\Bigg(- x^2 \cos(x) + 2 I_3 \Bigg) \\
&= x^3 \sin(x) - 3\Bigg(- x^2 \cos(x) + 2\Big(x \sin(x) + \cos(x) \Big) \Bigg) + C \\
&= x^3 \sin(x) + 3x^2 \cos(x) - 6x \sin(x) - 6 \cos(x) + C
\end{aligned}
$$