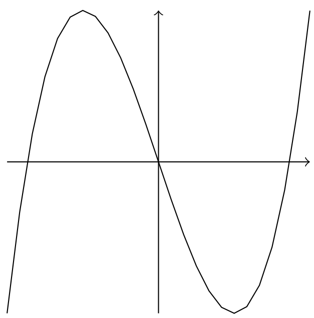
Math 8

Fall 2019

Sample Solutions to Practice Problems for Exam II

1. The curve in the picture is the graph of the function $y = x^3 - 3x$ in the xy-plane. The picture includes the region of the plane $-2 \leq x \leq 2$, $-2 \leq y \leq 2$.



(a) Give a function $\vec{r}(t)$ parametrizing this curve.

Solution: $\vec{r}(t) = \langle t, t^3 - 3t\rangle$. (This parametrizes the curve oriented in the direction from left to right.)

(This is only one of many possibilities. For example, $\vec{p}(t) = \langle -t, -t^3 + 3t\rangle$ parametrizes the same curve, but oriented from right to left.)

(b) For your function $\vec{r}$, find $\displaystyle\int_0^1 \vec{r}\,'(t)\,dt$.

Solution: $\displaystyle\int_0^1 \vec{r}\,'(t)\,dt = \vec{r}(1) - \vec{r}(0) = \langle 1, -2\rangle$.

(c) *Without calculating the unit tangent vector at all points*, find all points on the curve at which the unit tangent vector $\vec{T}$ is parallel to the x-axis. For your parametrization, what is the value of $\vec{T}$ at those points?

Solution: $\vec{T}$ is in the same direction as $\vec{r}\,'(t) = \langle 1, 3t^2 - 3\rangle$, and that is parallel to the x-axis when $3t^2 - 3 = 0$, or when $t = \pm 1$, at the points $(-1, 2)$ and $(1, -2)$. For this parametrization, $\vec{T} = \langle 1, 0\rangle$ at those points. (For a parametrization orienting the curve from right to left, we would have $\vec{T} = \langle -1, 0\rangle$.)

Alternatively, $\vec{T}$ is parallel to the x-axis exactly when the graph $y = x^3 - 3x$ has a horizontal tangent line, which is when the derivative $\dfrac{dy}{dx}$ equals zero.

(d) *Without calculating the unit normal vector at all points*, find all points on the curve at which the unit normal vector $\vec{N}$ is equal to $\langle 0, 1\rangle$.

Explain your reasoning.

Solution: Since $\vec{N} \perp \vec{T}$, the only points at which we can have $\vec{N} = \langle 0, 1 \rangle$ are the two points we found above. Also, $\vec{N}$ points in the direction in which the curve is bending, which is downward at $(-1, 2)$ and upward at $(1, -2)$. Therefore, $\vec{N} = \langle 0, 1 \rangle$ only at $(1, -2)$.

(e) Assuming your parametrization is the position function of a moving particle, find the tangential and normal components of the acceleration when the particle is at the point $(1, -2)$.

These are scalars $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ for which we write acceleration as $\vec{a} = a_{\mathbf{T}}\vec{T} + a_{\mathbf{N}}\vec{N}$.

Solution: The acceleration is $\vec{r}\,''(t) = \langle 0, 6t \rangle$. At the point $(-1, 2)$ this is perpendicular to the velocity (by part c), so the tangential component of the acceleration is 0 and the normal component of the acceleration is the magnitude of the acceleration, or, since $t = 1$ at this point, 6.

(f) When the particle is at the point $(1, -2)$, is its speed increasing, decreasing, or neither?

Solution: Since the tangential component of the acceleration is zero, the speed is neither increasing nor decreasing.

(g) Write down an integral that gives the arc length of the portion of this curve where $-1 \le x \le 1$. Do not evaluate this integral.

Solution: The speed is $|\vec{r}\,'(t)| = \sqrt{9t^4 - 18t^2 + 10}$, so the arc length is $\displaystyle\int_{-1}^{1} \sqrt{9t^4 - 18t^2 + 10}\, dt.$

2. The curve γ is the intersection of the surfaces $x^2 - 2x + 4y^2 + 16y = -13$ and $x + 2y - z = 2$.

(a) Find a function parametrizing γ.

(b) Find the curvature of γ at the point $(1, -1, -3)$.

Solution:

(a) Complete the square in the first equation to get $(x-1)^2 + (2y+4)^2 = 4$, or $\left(\dfrac{x-1}{2}\right)^2 + (y+2)^2 = 1$. Set $\dfrac{x-1}{2} = \cos t$ and $y + 2 = \sin t$, or $x = 2\cos t + 1$ and $y = \sin t - 2$. Use the second equation to solve for z, getting $z = x + 2y - 2 = 2\cos t + 2\sin t - 5$:

$$\vec{r}(t) = \langle 2\cos t + 1, \sin t - 2, 2\cos t + 2\sin t - 5 \rangle .$$

(b) The point $(1, -1, -3)$ corresponds to $t = \dfrac{\pi}{2}$, when $\cos t = 0$ and $\sin t = 1$.

We have $\vec{r}'(t) = \langle -2\sin t, \cos t, -2\sin t + 2\cos t\rangle$ so $\vec{r}'\left(\frac{\pi}{2}\right) = \langle -2, 0, -2\rangle$, and $\vec{r}''(t) = \langle -2\cos t, -\sin t, -2\cos t - 2\sin t\rangle$ so $\vec{r}''\left(\frac{\pi}{2}\right) = \langle 0, -1, -2\rangle$.

Now we have

$$\kappa = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{|\langle -2, 0, -2\rangle \times \langle 0, -1, -2\rangle|}{|\langle -2, 0, -2\rangle|^3} = \frac{|\langle -2, 4, -2\rangle|}{(2\sqrt{2})^3} = \frac{\sqrt{3}}{8}$$

Alternatively, we can find the components of the acceleration. We have

$$a_{\mathbf{T}}\vec{T} = proj_{\vec{v}}(\vec{a}) = \frac{\vec{a}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{\langle 0, -1, -2\rangle \cdot \langle -2, 0, -2\rangle}{\langle -2, 0, -2\rangle \cdot \langle -2, 0, -2\rangle}\langle -2, 0, -2\rangle = \langle -1, 0, -1\rangle$$

$$a_{\mathbf{N}}\vec{N} = \vec{a} - a_{\mathbf{T}}\vec{T} = \langle 0, -1, -2\rangle - \langle -1, 0, -1\rangle = \langle 1, -1, -1\rangle$$

$$\left(\frac{ds}{dt}\right)^2 \kappa = a_{\mathbf{N}} = \left|a_{\mathbf{N}}\vec{N}\right| = |\langle 1, -1, -1\rangle| = \sqrt{3}.$$

$$\frac{ds}{dt} = |\vec{r}'| = |\langle -2, 0, -2\rangle| = \sqrt{8}.$$

Therefore,

$$\kappa = \frac{a_{\mathbf{N}}}{\left(\frac{ds}{dt}\right)^2} = \frac{\sqrt{3}}{8}$$

3. TRUE or FALSE? Explain briefly.

   Note: If a statement is true, your explanation could be that it is a definition or a theorem, something that was explicitly stated in the textbook. Otherwise, you should give a brief explanation of why we know it is true.

   If a statement is false, your explanation could be an explanation of why it is not true, an example of a specific case when it is false, or a correction to make it a true statement.

   (a) Suppose a curve γ is defined as the intersection of two given surfaces. If the curvature κ for γ is computed using two different parametrizations, the answer will always be the same.

   TRUE. This is a theorem.

   (b) If $\vec{r}(t)$ parametrizes any curve on the sphere $x^2 + y^2 + z^2 = 1$, then the derivative $\vec{r}'(t)$ is always normal to the position vector $\vec{r}(t)$.

   TRUE. If $|\vec{r}|$ is constant, then $\vec{r} \perp \vec{r}'$. This is a theorem.

   (c) The curve parametrized by $\vec{r}(t) = \langle t^3, t^3, t\rangle$ is not a smooth curve.

   FALSE. This is a smooth parametrization, since $\vec{r}'(t) = \langle 3t^2, 3t^2, 1\rangle$ is never zero.

(d) If a particle travels around a circle, then the normal component of its acceleration is constant.

FALSE. The curvature κ is constant, but the normal component of acceleration is $\left(\frac{ds}{dt}\right)^2 \kappa$, which will change if the particle's speed changes.

(e) Two planes parallel to the same line are parallel.

FALSE. The xz-plane and yz-plane are both parallel to the z-axis, but they are not parallel to each other.

(f) Two planes perpendicular to the same plane are parallel.

FALSE. The xz-plane and yz-plane are both perpendicular to the xy-plane, but they are not parallel.

(g) If $\vec{r(t)}$ is the position function for a particle moving in the plane $x + y + z = 3$, then the velocity vector $\vec{v(t)}$ is always normal to the vector $\langle 1, 1, 1\rangle$.

TRUE. The velocity vector is tangent to the path, so parallel to the plane, so normal to its normal vector.

(h) If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are nonzero vectors such that $\vec{u} \cdot \vec{v} = 0$ and $\vec{u} \times \vec{w} = \vec{0}$, then $\vec{v} \cdot \vec{w} = 0$.

TRUE. From $\vec{u} \cdot \vec{v} = 0$ we see $\vec{u}$ and $\vec{v}$ are perpendicular, and from $\vec{u} \times \vec{w} = \vec{0}$ we see $\vec{u}$ and $\vec{w}$ are parallel. Therefore, $\vec{w}$ and $\vec{v}$ are also perpendicular.

4. Find the vector $(\vec{i} + \vec{j}) \times (\vec{i} - \vec{j})$.

Solution: We can do this either algebraically or geometrically. Geometrically, $\vec{i} + \vec{j}$ and $\vec{i} - \vec{j}$ are normal to each other, so the sine of the angle between them is 1, and each has magnitude $\sqrt{2}$, so the magnitude of their cross product will be the product of their magnitudes, or 2.

Since they lie in the xy-plane, the cross product is in the direction of either $\vec{k}$ or $-\vec{k}$. Drawing them and using the right hand rule gives us $-\vec{k}$.

Therefore the answer is $\langle 0, 0, -2\rangle$.

5. Find the center and radius of the sphere with equation $x^2 + y^2 + z^2 - 2x + 4z = 164$.

Solution: Complete the square:

$$x^2 - 2x + 1 - 1 + y^2 + z^2 + 4z + 4 - 4 = 164$$

$$(x - 1)^2 + y^2 + (z + 2)^2 = 169 = (13)^2.$$

The center is $(1, 0, -2)$ and the radius is 13.

6. Find the equation of the line perpendicular to the plane $x + 3y + z = 5$ and passing through $(1, 0, 6)$.

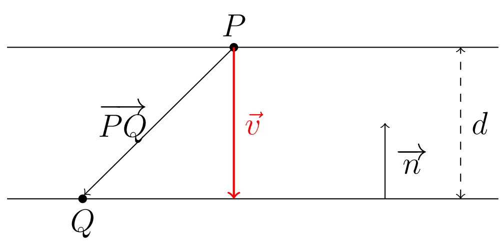
Solution: A normal vector to the plane, which gives the direction of the line, is $\langle 1, 3, 1\rangle$. (The coordinates come from the coefficients of x, y, and z in the equation for the plane.) An equation for the line is

$$\langle x, y, z\rangle = \langle 1, 0, 6\rangle + t\langle 1, 3, 1\rangle.$$

7. Find the distance between the planes with equations $2x-3y+z = 4$ and $4x-6y+2z = 3$. If the distance is 0, find the angle between the planes.

Solution: We can see from the equations that these two planes are parallel and not the same plane, so we must find the distance between them.

Here is one of many possible methods. The picture shows the two planes, a normal vector $\vec{n}$, points P and Q on the planes, and the distance d between the planes.



From the picture, the distance d we want is the length of the vector $\vec{v}$. This vector is the projection of $\overrightarrow{PQ}$ in the direction of $\vec{n}$, and its length is $\left|\text{comp}_{\vec{n}}(\overrightarrow{PQ})\right| = \dfrac{|\overrightarrow{PQ}\cdot\vec{n}|}{|\vec{n}|}$.

We can find a normal vector from the equations of the planes, say $\vec{n} = \langle 2, -3, 1\rangle$. We can find points P and Q on the planes by setting $y = z = 0$ and using the equations of the planes to solve for x, giving $P = (2, 0, 0)$ and $Q = \left(\dfrac{3}{4}, 0, 0\right)$, so $\overrightarrow{PQ} = \left\langle \dfrac{-5}{4}, 0, 0\right\rangle$.

Now

$$d = \frac{|\overrightarrow{PQ}\cdot\vec{n}|}{|\vec{n}|} = \left|\frac{\frac{-10}{4}}{\sqrt{14}}\right| = \frac{5\sqrt{14}}{28}.$$

Note: If you don't remember the formula for the component of one vector in the direction of another, you can use the geometry of the situation to see that $d = \left|\overrightarrow{PQ}\right| |\cos\theta|$, where θ is the angle between $\overrightarrow{PQ}$ and the normal vector $\vec{n}$, and then find $\cos\theta$ using the dot product.

Here is another method: Find an equation for the line through the origin normal to both planes:

$$\langle x, y, z\rangle = t\langle 2, -3, 1\rangle = \langle 2t, -3t, t\rangle.$$

Find the points where this meets the two planes. For the first plane:

$$2(2t) - 3(-3t) + t = 4 \qquad 14t = 4 \qquad t = \frac{2}{7} \qquad \langle x, y, z\rangle = \langle 2t, -3t, t\rangle = \left\langle \frac{4}{7}, \frac{-6}{7}, \frac{2}{7} \right\rangle.$$

For the second plane:

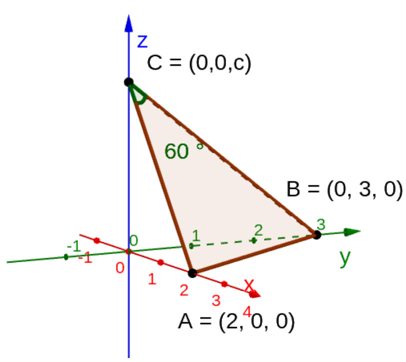
$$4(2t) - 6(-3t) + 2t = 3 \qquad 28t = 3 \qquad t = \frac{3}{28} \qquad \langle x, y, z\rangle = \langle 2t, -3t, t\rangle = \left\langle \frac{6}{28}, \frac{-9}{28}, \frac{3}{28} \right\rangle.$$

Now find the distance between these two points. This is the length of the displacement vector between them,

$$\left| \left\langle \frac{4}{7}, \frac{-6}{7}, \frac{2}{7} \right\rangle - \left\langle \frac{6}{28}, \frac{-9}{28}, \frac{3}{28} \right\rangle \right| = \left| \left\langle \frac{16}{28}, \frac{-24}{28}, \frac{8}{28} \right\rangle - \left\langle \frac{6}{28}, \frac{-9}{28}, \frac{3}{28} \right\rangle \right| =$$

$$\left| \left\langle \frac{10}{28}, \frac{-15}{28}, \frac{5}{28} \right\rangle \right| = \frac{5}{28} | \langle 2, -3, 1\rangle | = \frac{5\sqrt{14}}{28}$$

8. What value of c makes this figure possible?



Solution: We have $\overrightarrow{CA} = \langle 2, 0, -c\rangle$ and $\overrightarrow{CB} = \langle 0, 3, -c\rangle$. Their dot product is c^2. Also, their dot product is the product of their magnitudes with the cosine of $60°$, which is $\sqrt{4+c^2}\sqrt{9+c^2}\left(\frac{1}{2}\right) = \frac{\sqrt{c^4+13c^2+36}}{2}$. We set $c^2 = \frac{\sqrt{c^4+13c^2+36}}{2}$ and solve for c. (We also use the fact that $c > 0$, and the quadratic formula.)

$$\begin{aligned}
c^2 &= \frac{\sqrt{c^4+13c^2+36}}{2} \\
4c^4 &= c^4 + 13c^2 + 36 \\
3c^4 &- 13c^2 - 36 = 0 \\
c^2 &= \frac{13 \pm \sqrt{169+432}}{6} = \frac{13+\sqrt{601}}{6} \\
c &= \sqrt{\frac{13+\sqrt{601}}{6}}.
\end{aligned}$$

9. An object is traveling counterclockwise along the circle $x^2 + y^2 = 1$, and its speed at time t is $t^2 + 1$. At time $t = 1$ the object is located at the point $(0, 1)$. Find its acceleration at time $t = 1$.

   Hint: First find the tangential and normal components of the acceleration.

   Solution: By the geometry of the situation, the unit tangent vector and unit normal vector to the object's path at time $t = 1$ are $\vec{T} = \langle -1, 0\rangle$ and $\vec{N} = \langle 0, -1\rangle$. The curvature of a circle of radius 1 is $\kappa = 1$.

   The tangential component of acceleration is the linear acceleration,

$$a_{\mathbf{T}} = \frac{d^2s}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d}{dt}(t^2+1) = 2t$$

   and at time $t = 1$ we have $a_{\mathbf{T}} = 2$. The normal component of the acceleration is

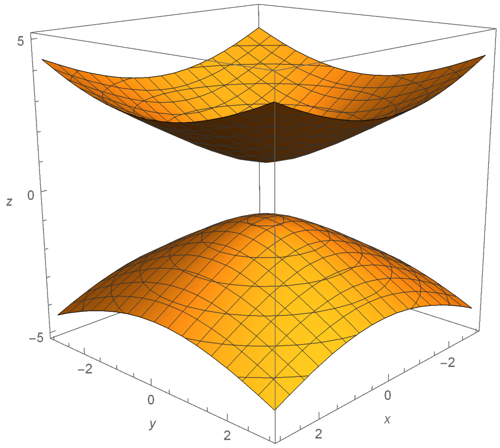
$$a_{\mathbf{N}} = \left(\frac{ds}{dt}\right)^2 (\kappa) = (t^2+1)^2(1) = (t^2+1)^2$$

   and at time $t = 1$ we have $a_{\mathbf{N}} = 4$.

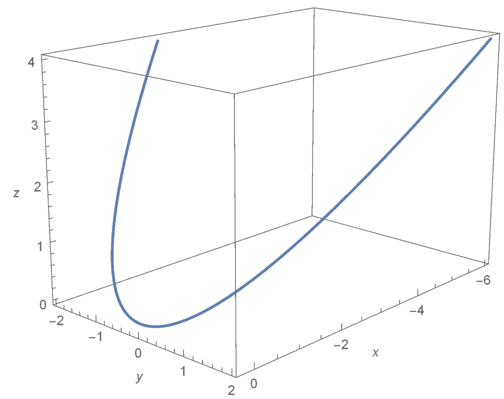
   Therefore the acceleration at time $t = 1$ is

$$\vec{a} = a_{\mathbf{T}}\vec{T} + a_{\mathbf{N}}\vec{N} = 2\langle -1, 0\rangle + 4\langle 0, -1\rangle = \langle -2, -4\rangle .$$

10. By the number of each picture, write the letter of the matching description, equation, or parametrization. The descriptions, the last two pictures, and the space for your answers, are on the next page.



1)



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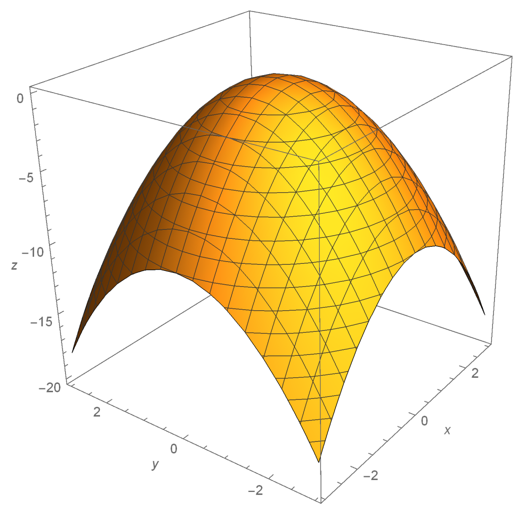
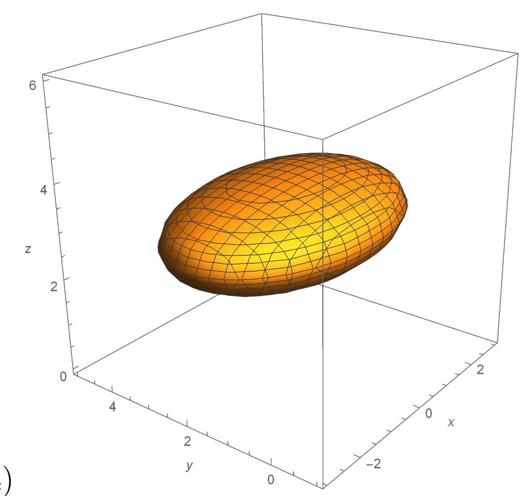
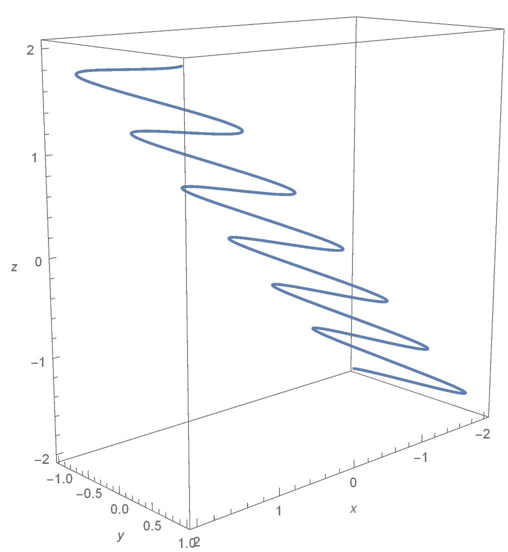


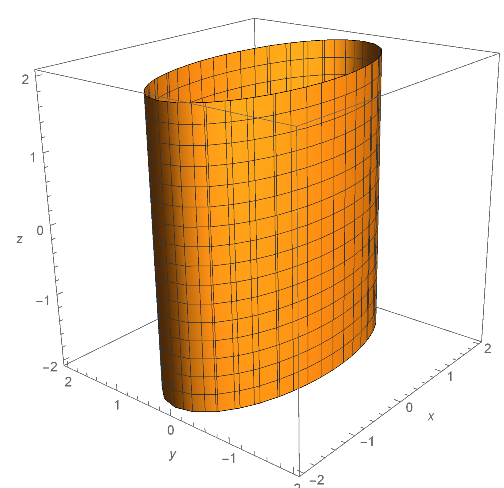
3)



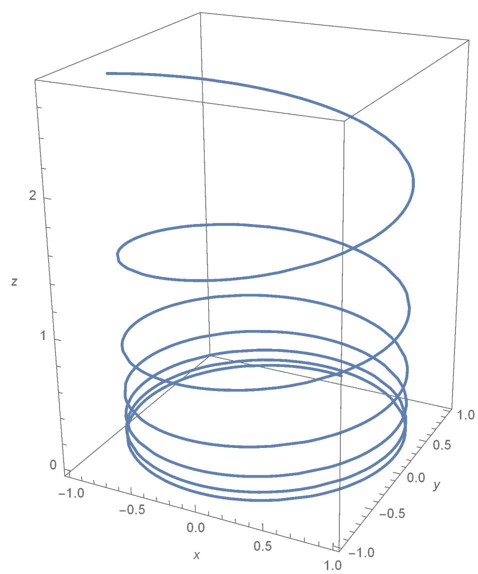
4)



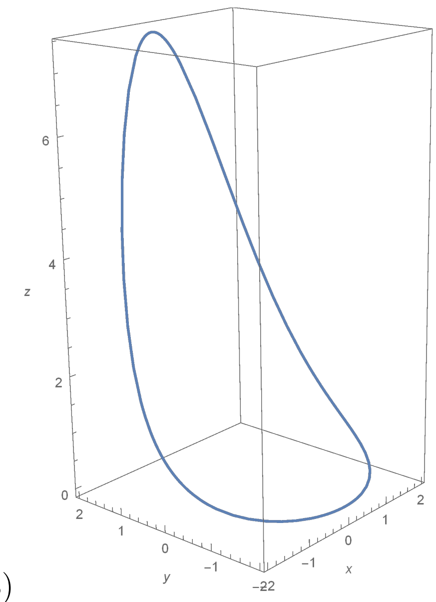
5)



6)



7)



8)

A $x^2 + y^2 + z = 0$

B $z^2 = 1 + x^2 + y^2$

C $z^2 + x^2 = 2 - y^2$

D $\frac{x^2}{9} + \frac{(y-2)^2}{4} + (z-3)^2 = 1$

E $4x - \frac{y}{3} + z = 20$

F $\frac{x^2}{4} + y^2 = 1$

G The intersection of $x = z$ and $y = \sin(x)$

H The intersection of $x^2 + y^2 = 2$ and $z = e^y$

I The intersection of $y^2 = z$ and the plane containing the origin and normal to the vector $\vec{v} = \langle 1, 1, 1 \rangle$

J The intersection of $x = z$ and $z = y^2$

K $\vec{r}(t) = \langle \cos(t), \sin(t), e^t \rangle$

L $\vec{r}(t) = \langle \cos(t), \sin(3t), t \rangle$

WRITE YOUR ANSWERS BELOW

Write by the number of each picture the letter of the matching description, equation, or parametrization:

1) **B** 2) **I** 3) **A** 4) **D**

5) **G** 6) **F** 7) **K** 8) **H**