Math 8 Fall 2019

Final Exam Practice Problems

1. Find the limit or show it does not exist.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^2}$

2. A study of the effectiveness of foam insulation placed giant foam blocks, originally at a uniform temperature of 80 degrees, into a refrigerated room maintained at 0 degrees, and studied the function $f(x,t)$, the temperature at a depth x millimeters into the block after t minutes in the refrigerated room.

   In a written homework problem we determined that the partial derivative $f_t(x,t)$ represented the rate of change of temperature with respect to time, and since we expect the block to be cooling off, we expect this partial derivative to be negative. Likewise, we determined that the partial derivative $f_x(x,t)$ represented the rate of change of temperature with respect to depth into the block, and since we expect warmer temperatures at greater depth, we expect this partial derivative to be positive.

   Shortly after the block is placed into the room, the temperature distribution inside the block is changing from a uniform temperature distribution, in which all points are the same temperature, to a temperature distribution in which points deeper into the block are warmer than points on the surface. Which of the following does this tell us we should expect?

   (Recall that, for example $f_{xt} = (f_x)_t$ is the rate of change of the partial derivative f_x with respect to time.)

   Circle the correct answer:

   $$f_{xt}(x,t) < 0 \qquad f_{xt}(x,t) > 0 \qquad f_{tx}(x,t) < 0 \qquad f_{tx}(x,t) > 0.$$

3. Consider the following possible properties of a function $f(x,y)$:

   (A.) The plane $z = 2x - y$ is tangent to the graph of f at the point $(1,1,1)$.

   (B.) $\dfrac{\partial f}{\partial x}(1,1) = 2$, $\dfrac{\partial f}{\partial y}(1,1) = -1$, and $f(1,1) = 1$.

   (C.) The function f is differentiable at the point $(1,1)$.

   Which of the following are true? Circle all correct answers. Note that by, for example, $(A) \implies (B)$ we mean that if (A) is true then (B) *must* also be true.

   $$(A) \implies (B) \qquad (B) \implies (A)$$
   $$(A) \implies (C) \qquad (C) \implies (A)$$
   $$(B) \implies (C) \qquad (C) \implies (B)$$

4. Here are the values of the first several derivatives of a function f at $x = 3$.

derivative	value at $x = 3$
$f(3) = f^{(0)}(3)$	7
$f^{(1)}(3)$	0
$f^{(2)}(3)$	0
$f^{(3)}(3)$	5
$f^{(4)}(3)$	2
$f^{(5)}(3)$	-1

(a) Write down the first three nonzero terms of the Taylor series of f centered at $x = 3$.

(b) Use part (a) to approximate the value of $f(2.8)$.

5. Give a function parametrizing the circle in the xy-plane with radius 3 and center $(-1, 2)$.

6. A thin wire is placed along the curve γ parametrized by the function $\vec{r}(t) = \langle t^2, t, t\rangle$ for $2 \leq t \leq 4$, where distance is measured in meters. The mass density of the wire at a point (x, y, z) on the wire is z kilograms per meter.

(a) Suppose that $\vec{r}(t)$ is a position function of a particle traveling along the wire. What is the speed of the particle at time $t = 3$? At time t in general?

(b) Find the approximate length of the portion of the wire between $\vec{r}(t)$ and $\vec{r}(t+\Delta t)$ if Δt is small. (This is the approximate distance traveled by the moving particle of part (a) between times t and $t + \Delta t$.)

(c) Find the approximate mass of the portion of the wire between $\vec{r}(t)$ and $\vec{r}(t+\Delta t)$.

(d) Write down a Riemann sum giving the approximate mass of the wire. Be sure to explain any variables you use. (If you use symbols like Δx, t_i, or n, what do they represent?)

(e) Write down an integral giving the mass of the wire.

7. The size of a washer is determined by three dimensions, its radius a, the radius b of the hole in the center, and its thickness c. The volume of the washer is given by $\pi a^2 c - \pi b^2 c$.

The standard sized washer for a certain application has dimensions $a = 8$, $b = 4$, and $c = .5$, all in millimeters (mm). One manufacturer's manufacturing tolerances allow a and b to differ from the standard by up to .1 mm, and c to differ from the standard by up to .05 mm.

In order to determine whether to pay more money to get better manufacturing tolerances, and therefore less variation in weight, an engineer must estimate how much the volume of a washer from this manufacturer could differ from the standard volume.

Use differentials to estimate how much the volume of a washer of approximate dimensions $a = 8$, $b = 4$, and $c = .5$, with an error of up to $\pm.1$ in a and b and up to $\pm.05$ in c, could differ from the volume of a washer of exact dimensions $a = 8$, $b = 4$, and $c = .5$.

8. The temperature at a point (x, y) in the plane is given by a function $f(x, y)$. The gradient of f is $\nabla f(x, y) = \langle x, -y \rangle$.

   (a) If the units of x and y are meters and the units of $f(x, y)$ are degrees, what are the units of the directional derivative $D_{\vec{u}}(x, y)$?

   (b) Find the directional derivative $D_{\vec{u}} f(3, 4)$ if $\vec{u}$ points from $(3, 4)$ directly toward the origin $(0, 0)$.

   (c) A moving object has position (x, y) at time t seconds given by $x = e^{2t}$ and $y = e^{-2t}$. Use the chain rule to find the speed (in degrees per second) at which the temperature experienced by the object is changing when $t = 0$.

   (The temperature experienced by the object means the temperature at the object's location.)

   (d) Show that at all times the object is moving in the direction in which temperature increases fastest.

9. Find the maximum value of the function $f(x, y, z) = 2xy + 2xz + 2yz$ subject to the constraint $x + y + z = 1$.

10. Find all the critical points of the function $f(x, y) = (x^2 + y^2)e^{-x}$, and determine for each whether it is a local minimum point, local maximum point, or saddle point.

11. Let D be the region in the xy-plane for which $0 \le x \le 1$ and $0 \le y < 1$. Note that this region is bounded but not closed.

    (a) Sketch the region D. Use solid lines to indicate edges that belong to D, and dotted lines to indicate edges that do not belong to D. Use filled circles to indicate corners that belong to D, and empty circles to indicate corners that do not belong to D.

    (b) Does $f(x, y) = xy$ have a minimum value on D? If so, what is that minimum value?

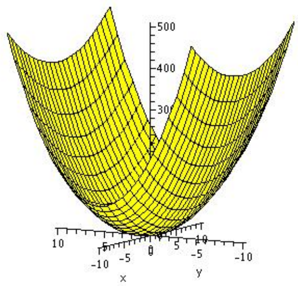
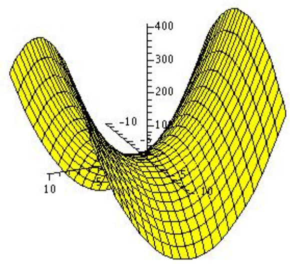
    (c) Does $f(x, y) = xy$ have a maximum value on D? If so, what is that maximum value?

12. Each function matches exactly one of the pictures, either a graph, or a set of level curves (for equally spaced values of f). Identify the picture that goes with each function.

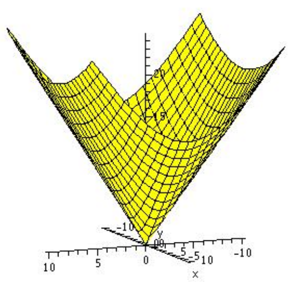
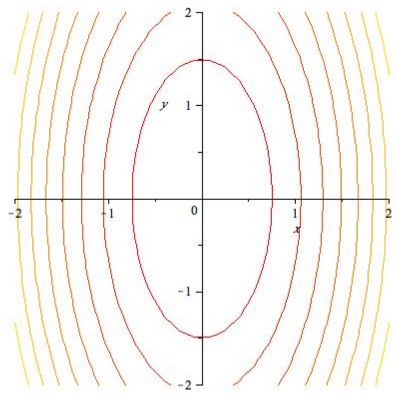
(a) $f(x,y) = x^2 + 4y^2$

(b) $f(x,y) = x^2 + 2xy + y^2$

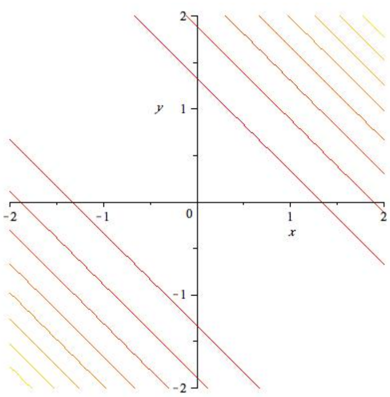
(c) $f(x,y) = 4x^2 - 2y^2$

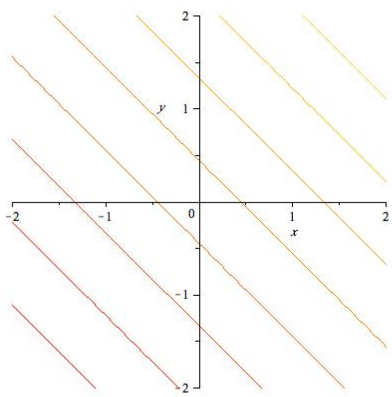



____________ **A** ______________________ **B** ____________________

____________ **C** ______________________ **D** ____________________





____________ **E** ______________________ **F** ____________________