Math 8 Fall 2019
Final Exam Practice Problems

1. Find the limit or show it does not exist.

(a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2} = \boxed{\text{DNE}}$

Solution: We can show this by showing two different ways (x, y) can approach $(0, 0)$ along which the function approaches different limits, or showing one way (x, y) can approach $(0, 0)$ along which the function has no limit.

For this example, as $(x, y) \to (0, 0)$ along the x-axis ($y = 0$), the value of the function is 0, so the limit approaching in that direction is 0. As $(x, y) \to (0, 0)$ along the line $y = x$, the value of the function is $\frac{3}{2}$, so the limit approaching in that direction is $\frac{3}{2}$.

A different way of saying this is that no matter how small a distance $\varepsilon > 0$ from $(0, 0)$ we choose, there are points (a, a) within that distance at which the function has value $\frac{3}{2}$, and other points $(a, 0)$ within that distance at which the function has value 0, so the function cannot possibly be approaching a limit.

(b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^2} = \boxed{0}$

Solution: One way to see this is to write x and y in polar coordinates, as $x = r\cos\theta$ and $y = r\sin\theta$. Since r is the distance from the origin, as $(x, y) \to (0, 0)$ we have $r \to 0$. Our function is

$$\frac{3xy^2}{x^2+y^2} = \frac{3r\cos\theta r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = 3r\cos\theta\sin^2\theta.$$

Since the values of cosine and sine are between -1 and 1, and r is positive, we have

$$-1 \le \cos\theta\sin^2\theta \le 1$$
$$-3r \le 3r\cos\theta\sin^2\theta \le 3r.$$

As $(x, y) \to (0, 0)$, we have $-3r \to 0$ and $3r \to 0$, so our function approaches 0. (You may notice that this is an application of the squeeze theorem.)

Another way is to write

$$\frac{3xy^2}{x^2+y^2} = \underbrace{3x}_{\to 0}\ \underbrace{\frac{y^2}{x^2+y^2}}_{0\le\frac{y^2}{x^2+y^2}\le 1}.$$

Since we can write our function as the product of a factor that is bounded (in this case between 0 and 1) and a factor that approaches 0, the function must approach 0.

2. A study of the effectiveness of foam insulation placed giant foam blocks, originally at a uniform temperature of 80 degrees, into a refrigerated room maintained at 0 degrees, and studied the function $f(x,t)$, the temperature at a depth x millimeters into the block after t minutes in the refrigerated room.

   In a written homework problem we determined that the partial derivative $f_t(x,t)$ represented the rate of change of temperature with respect to time, and since we expect the block to be cooling off, we expect this partial derivative to be negative. Likewise, we determined that the partial derivative $f_x(x,t)$ represented the rate of change of temperature with respect to depth into the block, and since we expect warmer temperatures at greater depth, we expect this partial derivative to be positive.

   Shortly after the block is placed into the room, the temperature distribution inside the block is changing from a uniform temperature distribution, in which all points are the same temperature, to a temperature distribution in which points deeper into the block are warmer than points on the surface. Which of the following does this tell us we should expect?

   (Recall that, for example $f_{xt} = (f_x)_t$ is the rate of change of the partial derivative f_x with respect to time.)

   Circle the correct answer:

   $$f_{xt}(x,t) < 0 \qquad \boxed{f_{xt}(x,t) > 0} \qquad f_{tx}(x,t) < 0 \qquad f_{tx}(x,t) > 0.$$

   Solution: As time goes on we are moving from a uniform distribution where temperature does not change as x changes, or $f_x(x,t) = 0$, to a distribution where deeper points are warmer, or $f_x(x,t) > 0$. This means that $f_x(x,t)$ is increasing with time, so its derivative with respect to time is positive: $f_{xt}(x,t) > 0$.

3. Consider the following possible properties of a function $f(x,y)$:

   (A.) The plane $z = 2x - y$ is tangent to the graph of f at the point $(1,1,1)$.

   (B.) $\dfrac{\partial f}{\partial x}(1,1) = 2$, $\dfrac{\partial f}{\partial y}(1,1) = -1$, and $f(1,1) = 1$.

   (C.) The function f is differentiable at the point $(1,1)$.

   Which of the following are true? Circle all correct answers. Note that by, for example, $(A) \implies (B)$ we mean that if (A) is true then (B) *must* also be true.

   $$\boxed{(A) \implies (B)} \qquad (B) \implies (A)$$

$$\boxed{(A) \implies (C)} \qquad (C) \implies (A)$$

$$(B) \implies (C) \qquad (C) \implies (B)$$

Solution: By the definition of "differentiable," a function that has a tangent plane is differentiable, so $\boxed{(A) \implies (C)}$. This arrow does not go the other way, because f could be differentiable but have a different tangent plane than the one given.

By the formula for tangent plane we learned, we have $\boxed{(A) \implies (B)}$. This arrow does not go the other way, because it is possible for a function to have partial derivatives but not be differentiable. It could be that (B) is true, but f is not differentiable, so f has no tangent plane at all.

The same reasoning tells us that $(B) \implies (C)$ is not true. Because f could be differentiable but have different partial derivatives than those given in (B), we see that $(C) \implies (B)$ is not true either.

4. Here are the values of the first several derivatives of a function f at $x = 3$.

derivative	value at $x = 3$
$f(3) = f^{(0)}(3)$	7
$f^{(1)}(3)$	0
$f^{(2)}(3)$	0
$f^{(3)}(3)$	5
$f^{(4)}(3)$	2
$f^{(5)}(3)$	-1

(a) Write down the first three nonzero terms of the Taylor series of f centered at $x = 3$.

Solution: $T_4(x) = 7 + \dfrac{5(x-3)^3}{3!} + \dfrac{2(x-3)^4}{4!} = \boxed{7 + \dfrac{5(x-3)^3}{6} + \dfrac{(x-3)^4}{12}}$

(b) Use part (a) to approximate the value of $f(2.8)$.

Solution: $f(2.8)$ is roughly

$$T_4(2.8) = 7 + \frac{5(2.8-3)^3}{6} + \frac{(2.8-3)^4}{12} = \boxed{7 - \frac{0.04}{6} + \frac{0.0016}{12}}$$

5. Give a function parametrizing the circle in the xy-plane with radius 3 and center $(-1, 2)$.

Solution: There is an infinite number of correct parametrizations, but the simplest is

$$\boxed{\vec{r}(t) = \langle 3\cos(t) - 1, 3\sin(t) + 2 \rangle.}$$

6. A thin wire is placed along the curve γ parametrized by the function $\vec{r}(t) = \langle t^2, t, t\rangle$ for $2 \le t \le 4$, where distance is measured in meters. The mass density of the wire at a point (x, y, z) on the wire is z kilograms per meter.

   (a) Suppose that $\vec{r}(t)$ is a position function of a particle traveling along the wire. What is the speed of the particle at time $t = 3$? At time t in general?

   Solution: The velocity is $\vec{r}\,'(t) = \langle 2t, 1, 1\rangle$ and the speed is $|\vec{r}\,'(t)| = \sqrt{4t^2 + 2}$. At $t = 3$ the particle's speed is $\boxed{\sqrt{38}\dfrac{\text{m}}{\text{sec}}}$ and in general it is $\boxed{\sqrt{4t^2+2}\dfrac{\text{m}}{\text{sec}}}$.

   (b) Find the approximate length of the portion of the wire between $\vec{r}(t)$ and $\vec{r}(t+\Delta t)$ if Δt is small. (This is the approximate distance traveled by the moving particle of part (a) between times t and $t + \Delta t$.)

   Solution: This distance can be approximated by the speed of the moving particle at time t multiplied by the length of time, or $\boxed{\sqrt{4t^2+2}(\Delta t)\text{m}}$. (This is an approximation because the particle's speed will change slightly during that time interval.)

   (c) Find the approximate mass of the portion of the wire between $\vec{r}(t)$ and $\vec{r}(t+\Delta t)$.

   Solution: The mass density at the point $\vec{r}(t) = \langle t^2, t, t\rangle$ is given by the z-coordinate, or t. To find the mass (in kilograms), multiply the mass density (in kilograms per meter) times the length (in meters). This is approximately $\boxed{t\sqrt{4t^2+2}(\Delta t)\text{kg}}$.

   (d) Write down a Riemann sum giving the approximate mass of the wire. Be sure to explain any variables you use. (If you use symbols like Δx, t_i, or n, what do they represent?)

   Solution: Divide the interval $2 \le t \le 4$ into n-many pieces of length Δt, and choose t_i in the i^{th} interval. Add up the approximate masses of the pieces:

$$\boxed{\text{mass} \approx \sum_{i=1}^{n} t_i\sqrt{4(t_i)^2+2}(\Delta t)\text{kg}}.$$

(e) Write down an integral giving the mass of the wire.

Solution: Take the limit of the Riemann sum as $n \to \infty$ to get

$$\boxed{\text{mass} = \int_2^4 t\sqrt{4t^2+2}\,dt\,\text{kg}}.$$

7. The size of a washer is determined by three dimensions, its radius a, the radius b of the hole in the center, and its thickness c. The volume of the washer is given by $\pi a^2 c - \pi b^2 c$.

   The standard sized washer for a certain application has dimensions $a = 8$, $b = 4$, and $c = .5$, all in millimeters (mm). One manufacturer's manufacturing tolerances allow a and b to differ from the standard by up to .1 mm, and c to differ from the standard by up to .05 mm.

   In order to determine whether to pay more money to get better manufacturing tolerances, and therefore less variation in weight, an engineer must estimate how much the volume of a washer from this manufacturer could differ from the standard volume.

   Use differentials to estimate how much the volume of a washer of approximate dimensions $a = 8$, $b = 4$, and $c = .5$, with an error of up to $\pm .1$ in a and b and up to $\pm .05$ in c, could differ from the volume of a washer of exact dimensions $a = 8$, $b = 4$, and $c = .5$.

   Solution: Volume V is a function of the dimensions a, b, and c. The differential is

$$dV = \frac{\partial V}{\partial a}da + \frac{\partial V}{\partial b}db + \frac{\partial V}{\partial c}dc = (2\pi ac)da - (2\pi bc)db + (\pi a^2 - \pi b^2)dc.$$

   We can use this to estimate the possible difference in volume from the standard washer with dimensions $a = 8$, $b = 4$, $c = .5$,

$$\Delta V \approx (2\pi ac)\Delta a - (2\pi bc)\Delta b + (\pi a^2 - \pi b^2)\Delta c = (8\pi)\Delta a - (4\pi)\Delta b + (48\pi)\Delta c.$$

   Given that

$$|\Delta a| \le .1 \qquad |\Delta b| \le .1 \qquad |\Delta c| \le .05$$

   we can estimate that

$$|\Delta V| \approx |(8\pi)(\Delta a) - (4\pi)\Delta b + (48\pi)\Delta c| \le$$

$$|(8\pi)(\Delta a)| + |(4\pi)\Delta b| + |(48\pi)\Delta c| \le 8\pi(.1) + 4\pi(.1) + 48\pi(.05) = 3.6\pi.$$

   The units of ΔV are cubic millimeters, so our estimate is $\boxed{3.6\pi \text{mm}^3}$.

8. The temperature at a point (x, y) in the plane is given by a function $f(x, y)$. The gradient of f is $\nabla f(x, y) = \langle x, -y \rangle$.

   (a) If the units of x and y are meters and the units of $f(x, y)$ are degrees, what are the units of the directional derivative $D_{\vec{u}}(x, y)$?

   Solution: Degrees per meter.

   (b) Find the directional derivative $D_{\vec{u}} f(3, 4)$ if $\vec{u}$ points from $(3, 4)$ directly toward the origin $(0, 0)$.

   Solution: $\nabla f(3, 4) = \langle 3, -4 \rangle$, and $\vec{u} = \left\langle \frac{-3}{5}, \frac{-4}{5} \right\rangle$, so the directional derivative is $D_{\vec{u}} f(3, 4) = \nabla f(3, 4) \cdot \vec{u} = \boxed{\frac{7}{5} \frac{\text{deg}}{\text{m}}}$.

   (c) A moving object has position (x, y) at time t seconds given by $x = e^{2t}$ and $y = e^{-2t}$. Use the chain rule to find the speed (in degrees per second) at which the temperature experienced by the object is changing when $t = 0$.

   (The temperature experienced by the object means the temperature at the object's location.)

   Solution: At $t = 0$ we have $x = e^{2t} = 1$ and $y = e^{-2t} = 1$, $\frac{dx}{dt} = 2e^{2t} = 2$ and $\frac{dy}{dt} = -2e^{-2t} = -2$. We also have $\frac{\partial f}{\partial x} = x = 1$ and $\frac{\partial f}{\partial y} = -y = -1$. This gives

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (1)(2) + (-1)(-2) = \boxed{4 \frac{\text{deg}}{\text{sec}}}.$$

   (d) Show that at all times the object is moving in the direction in which temperature increases fastest.

   Solution: The location of the object at time t is $x = e^{2t}$ and $y = e^{-2t}$. The direction in which temperature is increasing fastest is the direction of the gradient, $\nabla f(x, y) = \langle x, -y \rangle = \langle e^{2t}, -e^{-2t} \rangle$.

   The direction in which the object is moving is the direction of the velocity $\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \langle 2e^{2t}, -2e^{-2t} \rangle$.

   Since these vectors are positive scalar multiples of each other, they point in the same direction.

9. Find the maximum value of the function $f(x, y, z) = 2xy + 2xz + 2yz$ subject to the constraint $x + y + z = 1$.

   Solution: Here we use Lagrange Multipliers. The constraint is $g(x, y, z) = x+y+z = 1$, so $\nabla f = \langle 2y + 2z, 2x + 2z, 2x + 2y \rangle$ and $\nabla g = \langle 1, 1, 1 \rangle$. Then we have the following equations to solve:

$$\langle 2y+2z, 2x+2z, 2x+2y\rangle = \lambda\langle 1,1,1\rangle$$

$$x+y+z=1$$

Examining the first equation, we get $2y+2z = 2x+2z = 2x+2y$, and this leads us to $x=y=z$. Then plugging this into the constraint, we have $3x=1$, so $x=y=z=\frac{1}{3}$.

Finally, the maximum value is $\boxed{f\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)=\frac{2}{3}}$.

10. Find all the critical points of the function $f(x,y)=(x^2+y^2)e^{-x}$, and determine for each whether it is a local minimum point, local maximum point, or saddle point.

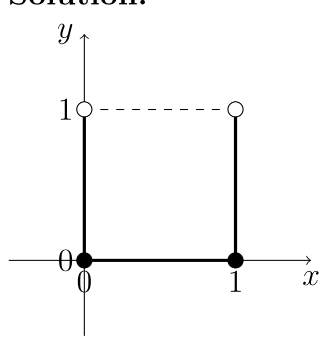
    Solution: For this, we need the partial derivatives $f_x(x,y)$ and $f_y(x,y)$. These are $f_x(x,y)=(2x-x^2-y^2)e^{-x}$ and $f_y(x,y)=2ye^{-x}$. The latter is equal to 0 only when $y=0$, and they are both equal to 0 whenever $y=0$ and $2x-x^2=0$, that is when $(x,y)=(0,0)$ or $(x,y)=(2,0)$. To find whether they are extrema, we use the second derivative test. Here, $f_{xx}=e^{-x}(2-2x-2x+x^2+y^2)$, $f_{xy}=-2ye^{-x}$ and $f_{yy}=2e^{-x}$.

    - $(2,0)$ is not a local extremum (maximum or minimum) point, since $f_{xx}(2,0)<0$ and $f_{yy}(2,0)>0$. It is a saddle point. (You can also check directly that the discriminant $f_{xx}f_{yy}-(f_{xy})^2$ is negative.)
    - $(0,0)$ is a local minimum point, since at $(0,0)$ we have $f_{xx}f_{yy}-(f_{xy})^2=4>0$ and $f_{yy}(0,0)=2>0$.

11. Let D be the region in the xy-plane for which $0\le x\le 1$ and $0\le y<1$. Note that this region is bounded but not closed.

    (a) Sketch the region D. Use solid lines to indicate edges that belong to D, and dotted lines to indicate edges that do not belong to D. Use filled circles to indicate corners that belong to D, and empty circles to indicate corners that do not belong to D.

    Solution:

(b) Does $f(x, y) = xy$ have a minimum value on D? If so, what is that minimum value?

Solution: Yes, the minimum value of f on D is 0. That value is attained at points $(x, 0)$ for $0 \le x \le 1$ and points $(0, y)$ for $0 \le y < 1$; that is, at all points on the x- or y-axis that belong to D. It is easy to see this is a minimum, because at every point of D we have $x \ge 0$ and $y \ge 0$, so $xy \ge 0$.

(c) Does $f(x, y) = xy$ have a maximum value on D? If so, what is that maximum value?

Solution: No. The values of f can get arbitrarily close to 1 on D, because for every positive $a < 1$ the point $(1, a)$ belongs to D, and $f(1, a) = a$. However, the values of f on D never reach 1, because for all $(x, y) \in D$ we have $0 \le x \le 1$ and $0 \le y < 1$, so $0 \le xy \le y < 1$.

If we form a slightly larger closed region E, by adding to D the top edge ($0 \le x \le 1$, $y = 1$), then f does have a maximum value on E. That value is 1, and it is attained at the point $(1, 1)$.

12. Each function matches exactly one of the pictures, either a graph, or a set of level curves (for equally spaced values of f). Identify the picture that goes with each function.

(a) $f(x, y) = x^2 + 4y^2$

(b) $f(x, y) = x^2 + 2xy + y^2$

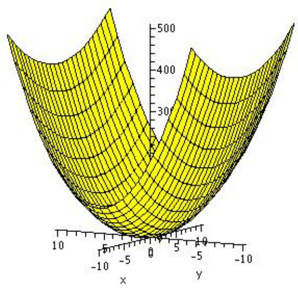
(c) $f(x, y) = 4x^2 - 2y^2$

Solution:

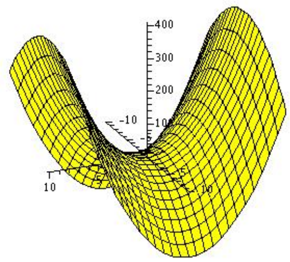
(a) $\boxed{A.}$ The intersection with the xz-plane is a parabola, and level curves are ellipses, ruling out B, C, E, F. Sketching the level curve $x^2 + 4y^2 = 1$ rules out D, since the ellipses are the wrong shape.

(b) $\boxed{E.}$ This is $(x + y)^2$, so the level curves are lines $x + y = c$, ruling out A, B, C, D. Sketching level curves for $f(x, y) = 0, 1, 2$ shows that the spacing rules out F.

(c) $\boxed{B.}$ The level curves are hyperbolae. (Alternatively, the intersections with the coordinate planes are an upward-facing parabola, a downward-facing parabola, and the crossed lines $y = \pm\sqrt{2}\,x$.)

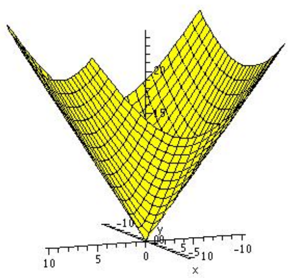


500
400
10
-10
-5
0
5
10
-5
-10
x
y

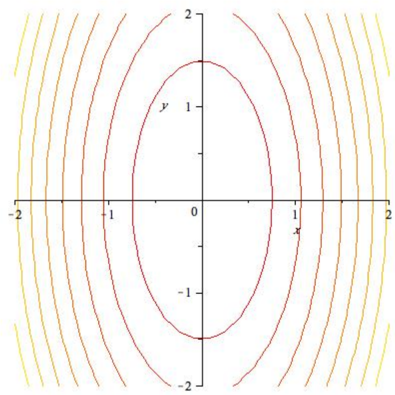


400
300
200
-10
10
10

______________ **A** __________________________ **B** ______________________

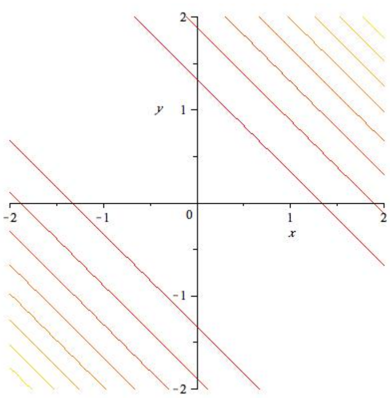


20
15
-10
10
5
0
5
10
-10
x
y

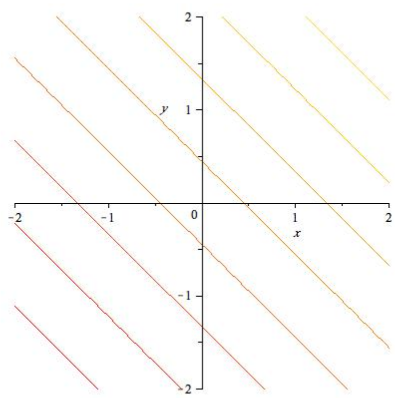


2
1
y
-2
-1
0
1
2
x
-1
-2

______________ **C** __________________________ **D** ______________________



2
1
y
-2
-1
0
1
2
x
-1
-2



2
1
y
-2
-1
0
1
2
x
-1
-2

______________ **E** __________________________ **F** ______________________