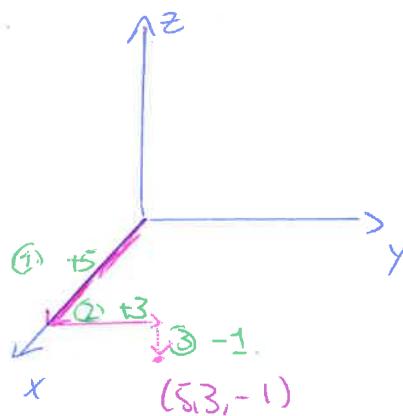
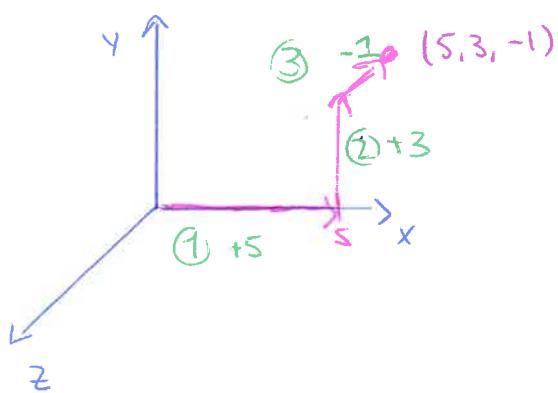


## 3D surfaces

## I - Coordinates



Where is  
 $(5, 3, -1)$ ?

Are they the same? We can rotate the projection of the space on the paper of  $120^\circ$ .

To draw a point:

The point  $(a, b, c)$  is drawn by moving from the origin:

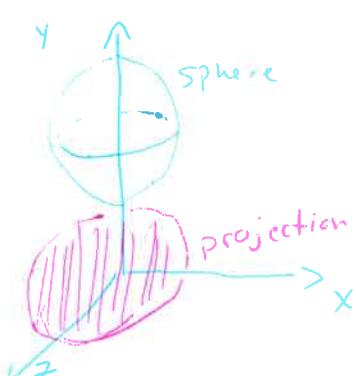
- ① one step of size  $a$  along the  $x$ -axis,
- ②  $b$  in the same direction as the  $y$ -axis, and
- ③  $-c$  in the direction of the  $z$ -axis.

## II - Projection.

The projection of a geometric object (point, line, surface, ...) onto a plane is a flattening of the object in the direction perpendicular to the plane.

Examples

A sphere onto the plane  $x=2$



For a point:

The point  $(a, b, c)$ , when projected on the  $yz$ -plane, is  $(0, b, c)$ .

What does the line (in dimension 3)  $x=2, y=3$  looks like on the plane perpendicular to the  $z$ -axis?

This plane is  $xy$ , and so the projection of  $\{(2, 3, z) | z \in \mathbb{R}\}$  is the point  $(2, 3)$ .

### III-Surfaces

An equation in  $x, y$  and  $z$  represents a surface in  $\mathbb{R}^3$ .

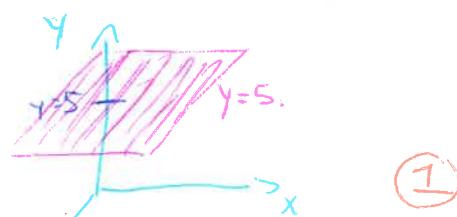
#### Example of surfaces

- plane
- sphere
- cylinder
- horse saddle
- ...

#### Examples

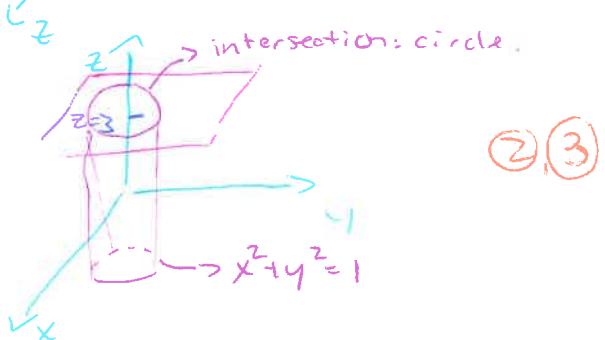
1- What is  $y=5$  in  $\mathbb{R}^3$ ?

A plane.



2- What is the intersection of the surfaces  $x^2+y^2=1$  and  $z=3$ ?

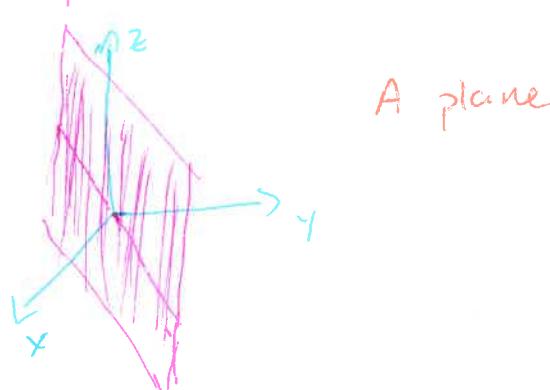
A circle.



3- What is  $x^2+y^2=1$ ?

A cylinder, of radius 1.

4- What is  $y=x$ ?



A plane

③

## IV - Distance in three dimensions

The distance  $|P_1 P_2|$  between two points  $P_1 = (x_1, y_1, z_1)$  and  $P_2 = (x_2, y_2, z_2)$  is

$$|P_1 P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

### Example

The distance between  $P(2, -1, 7)$  and  $Q(1, -3, 5)$  is

$$\begin{aligned}|PQ| &= \sqrt{(2-1)^2 + (-1-(-3))^2 + (7-5)^2} \\&= \sqrt{1+4+4} \\&= 3.\end{aligned}$$

## V Spheres.

A sphere is the set of all points at a given distance  $r$ , called the radius, from a point  $(h, k, l)$ , that is its center.

Its equation (as a surface in 3D) is

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

If the center is the origin  $(0, 0, 0)$ , the equation of a sphere of radius  $r$  is

$$x^2 + y^2 + z^2 = r^2$$

### Example.

Where is centered  $x^2 + y^2 + z^2 + 4x - 6y - 3 = 0$ ? What is its radius?

$$\begin{aligned}x^2 + y^2 + z^2 + 4x - 6y - 3 &= x^2 + 4x + 4 + y^2 - 6y + 9 + z^2 - 16 \quad (\text{completing the squares}) \\&\Rightarrow (x+2)^2 + (y-3)^2 + z^2 = 4^2.\end{aligned}$$

This is the equation of a sphere of radius 4 centered at  $(-2, 3, 0)$ .

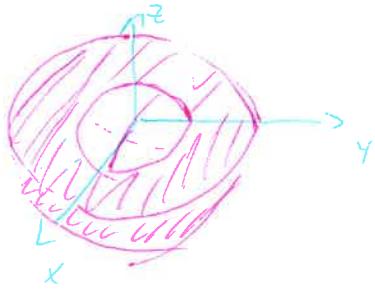
Example

What region of  $\mathbb{R}^3$  is represented by

$$1 \leq x^2 + y^2 + z^2 \leq 4?$$

What is its volume?

Solution: This is the region inside the sphere of radius 2, but outside the sphere of radius 1.



Its volume is the difference of volume of the two spheres:

$$\frac{4\pi(2)^3}{3} - \frac{4\pi(1)^3}{3} = \frac{28\pi}{3}.$$

Reference: James STEWART. Calculus, 8th edition.  
Section 12.1.