Math 8 - Lecture 12

Nadia Lafrenière

Dot product and projection

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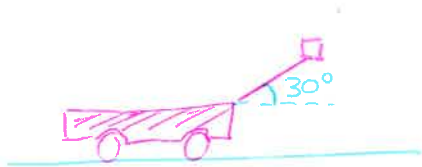
Problem: A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 30° above the horizontal. Find the work done by the force.



The problem is that the force is not applied in the same direction as the displacement.

Intermediate Step: What is the proportion of the force that is used for the displacement?

Orientation of the displacement: $\longrightarrow$

Orientation of the force: $\nearrow 30°$

The force is $\langle 70 \cdot \cos(30°), 70 \cdot \sin(70°) \rangle$ N, and only the first component counts for the work.

So, the work is

$$100 \cdot 70 \cdot \cos(30°) \approx 6062 \text{ J}$$

Here, we implicitly used the dot product, without knowing it.

Definition

If $\vec{u}$ and $\vec{v}$ are two vectors, their dot product is the real number defined as

$$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cdot \cos(\theta),$$

notation for dot product

Where θ is the angle between $\vec{u}$ and $\vec{v}$.

In the problem earlier, we computed the dot product of the displacement vector and the force vector.

Example

Compute the following dot products:

(a) $\langle -3,5\rangle \cdot \langle 5,3\rangle$

(b) $\langle -3,5\rangle \cdot \langle -3,5\rangle$

Solution

(a) In the first case, the two vectors are orthogonal (i.e. perpendicular to each other), and $\cos\left(\frac{\pi}{2}\right) = 0$.

(b) Here, we have the same vector twice, and the angle between them is 0. Hence,

$$|\langle -3,5\rangle|^2 \cdot \cos(0) = |\langle -3,5\rangle|^2 = 34$$

Properties of the dot product

(a) $\vec{u} \cdot \vec{v} = 0$ if and only if $\vec{u}$ and $\vec{v}$ are orthogonal ($\vec{0}$ is orthogonal to all vectors)

(b) $\vec{u} \cdot \vec{u} = |\vec{u}|^2$.

(c) $c \cdot \vec{u} \cdot \vec{v} = (c \cdot \vec{u}) \cdot \vec{v} = \vec{u} \cdot (c \cdot \vec{v})$, for $c \in \mathbb{R}$.

⚠ $c\ \vec{u} \cdot \vec{v} \neq (c \cdot \vec{u}) \cdot (c \cdot \vec{v})$

It is equivalent to multiplying the length of one vector.

(d) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

We will talk about it once we know the algebraic notation.

Algebraic notation

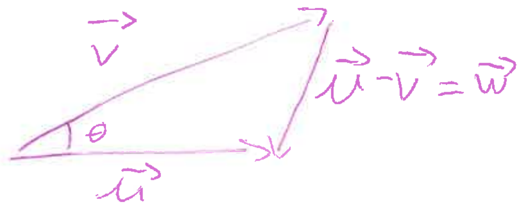
Proposition

If $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$ are vectors, then

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Proof

We have two 3-dimensional vectors $\vec{u}$ and $\vec{v}$, so there exists a plane (ie. a 2-dimensional object), that contains both, so we can draw them like this:



This is a triangle, so we can apply the Law of Cosines:

$$|\vec{w}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2\underbrace{|\vec{u}||\vec{v}|\cos(\theta)}_{\vec{u}\cdot\vec{v}}$$

Hence,

$$2\,\vec{u}\cdot\vec{v} = |\vec{u}|^2 + |\vec{v}|^2 - |\underbrace{\vec{u}-\vec{v}}_{\vec{w}}|^2$$

$$= (u_1^2+u_2^2+u_3^2)+(v_1^2+v_2^2+v_3^2)-\left((u_1-v_1)^2+(u_2-v_2)^2+(u_3-v_3)^2\right)$$

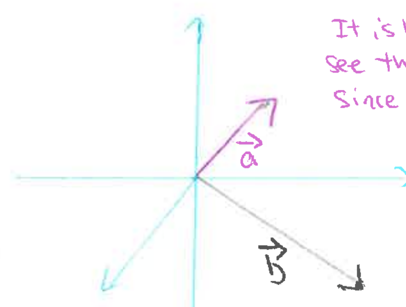
$$= 2u_1v_1 + 2u_2v_2 + 2u_3v_3.$$

thus,

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3. \quad \blacksquare$$

Using the latter definition of $\vec{u}\cdot\vec{v}$, it should be easy to understand why property (d) is true.

Problem: Find the angle between the vectors $\vec{a} = \langle 2,2,-1\rangle$ and $\vec{b} = \langle 5,-3,2\rangle$.



It is hard to see the angle, since it is in 3D.

Solution: On the one hand, we have

$$\vec{a}\cdot\vec{b} = 2\cdot 5 + 2\cdot(-3) + (-1)\cdot 2 = 2$$

On the other hand,

$$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos(\theta) = 3\cdot\sqrt{38}\cdot\cos\theta,$$

so

$$\theta = \arccos\left(\frac{2}{3\sqrt{38}}\right) = 83{,}79^\circ$$

Projection

The dot product is the key to find the projections of a vector onto another.

Definition

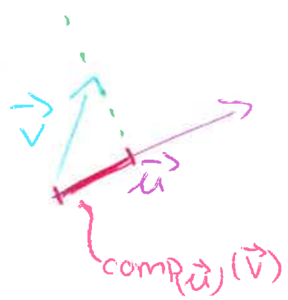
Given two vectors $\vec{u}$ and $\vec{v}$, the vector projection of $\vec{v}$ onto $\vec{u}$ is the part of $\vec{v}$ in the orientation of $\vec{u}$, and is given by

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\,\vec{u}$$

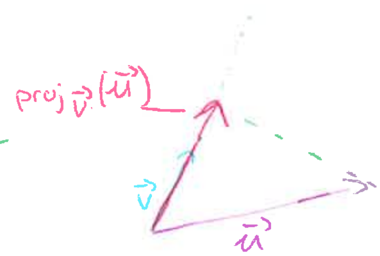
The scalar projection of $\vec{v}$ onto $\vec{u}$ is the signed magnitude of the projection (i.e. $\pm$ the length). It is also called the component along $\vec{v}$, and is

$$\text{comp}_{\vec{u}}(\vec{v}) = \pm|\text{proj}_{\vec{u}}(\vec{v})| = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|} = |\vec{v}|\cdot\cos(\theta),$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.



is the vector projection. Its length is $|\text{comp}_{\vec{u}}(\vec{v})|$.



Example

Find the vector and scalar projections of $\langle 1,1,2\rangle$ onto $\langle -2,3,1\rangle$

$\langle 1,1,2\rangle \cdot \langle -2,3,1\rangle = 3$, $\quad |\langle -2,3,1\rangle| = \sqrt{14}$

Then,

$\text{comp}_{\langle -2,3,1\rangle}(\langle 1,1,2\rangle) = \frac{3}{\sqrt{14}}$ and $\text{proj}_{\langle -2,3,1\rangle}(\langle 1,1,2\rangle) = \frac{3}{14}\langle -2,3,1\rangle = \langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\rangle$.

Reference: James Stewart. Calculus, eighth edition. §12.3.