

Cross product.

10/14/2019

Problem

Let \vec{u} , \vec{v} and \vec{w} be vectors.

How do you know if there exists scalars (i.e. real numbers) k, l so that $k\vec{u} + l\vec{v} = \vec{w}$?

Examples for the 3-dim. vectors:
 $\vec{u} = \langle 1, 3, 2 \rangle$, $\vec{v} = \langle 0, 5, 8 \rangle$, $\vec{w} = \langle 3, 3, 3 \rangle$ we
 $\vec{u} = \langle 1, 4, -7 \rangle$, $\vec{v} = \langle 2, -1, 4 \rangle$, $\vec{w} = \langle 0, -1, 2 \rangle$: $k=2/9$, $l=1/4$

- If those vectors are two-dimensional and \vec{u} and \vec{v} are not collinear (i.e. parallel), any vector \vec{w} can be written in that way. Said otherwise, \vec{u} and \vec{v} generate the whole plane.
- If two 3-dimensional vectors are not collinear, they generate a plane, so it depends on the fact that \vec{w} is in that plane.
(Give the examples above.)

How to do this?

We build a parallelepiped (a 3D analogue of a parallelogram), whose sides are \vec{u} , \vec{v} and \vec{w} . If \vec{u} , \vec{v} and \vec{w} are in the same plane, the solid is flat and its volume is 0 .

The volume of this solid is computed by the determinant of \vec{u} , \vec{v} and \vec{w}

Definition (2D version)

Let $\vec{u} = \langle a, b \rangle$ and $\vec{v} = \langle c, d \rangle$, (2-dimensional vectors)

The determinant of \vec{u} and \vec{v} is the (signed) area

of the parallelogram generated by \vec{u} and \vec{v} . It is positive if going from \vec{u} to \vec{v} creates an angle between 0° and 180° .

It is computed using

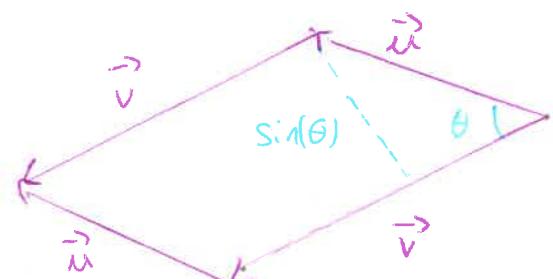
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = |\vec{u}| |\vec{v}| \sin(\theta).$$



Example: $\vec{u} = \langle 3, 1 \rangle$, $\vec{v} = \langle -4, -2 \rangle$

The area is

$$\begin{vmatrix} 3 & 1 \\ -4 & -2 \end{vmatrix} = (-3)(-2) - (-1)(1) = 10 \text{ units}^2$$



Definition

(2)

let $\vec{u} = \langle a, b, c \rangle$, $\vec{v} = \langle d, e, f \rangle$, $\vec{w} = \langle g, h, i \rangle$ be 3-dimensional vectors.

The determinant of \vec{u} , \vec{v} and \vec{w} is the (signed) volume of the parallelepiped generated by \vec{u} , \vec{v} and \vec{w} , and is computed by

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ = a(iei-hf) - b(di-gf) + c(dh-ge).$$

Tip to remember the formula:

- The numbers that multiply the 2D determinant are the numbers in the first row: $a, -b, c$, with the signs alternating.
- The determinant that is multiplied by a is that obtained by deleting the row and column that contain a . It is the same thing for b and c .

The orientation is given by the right-hand rule (we will talk about it soon).

Example

- The vectors $\langle -1, 3, 2 \rangle$, $\langle 0, 5, 8 \rangle$, $\langle 3, 3, 3 \rangle$ have determinant

$$\begin{vmatrix} -1 & 3 & 2 \\ 0 & 5 & 8 \\ 3 & 3 & 3 \end{vmatrix} = -1 \begin{vmatrix} 5 & 8 \\ 3 & 3 \end{vmatrix} - 3 \begin{vmatrix} 0 & 8 \\ 3 & 3 \end{vmatrix} + 2 \begin{vmatrix} 0 & 5 \\ 3 & 3 \end{vmatrix} \\ = (-1)(-9) - 3(-24) + 2(-15) \\ = 9 + 72 - 30 = 51$$

Hence, the three vectors do not lie in the same plane (otherwise their volume would be 0), and there is no solution to

$$k \cdot \langle -1, 3, 2 \rangle + l \cdot \langle 0, 5, 8 \rangle = \langle 3, 3, 3 \rangle, k, l \in \mathbb{R}.$$

Example

The vectors $\vec{u} = \langle 1, 4, -7 \rangle$, $\vec{v} = \langle 2, -1, 4 \rangle$ and $\vec{w} = \langle 0, -1, 2 \rangle$ satisfy

$$\frac{-2}{9} \vec{u} + \frac{1}{9} \vec{v} = \vec{w}.$$

Hence, the parallelepiped generated by them is flat, and has volume 0:

$$\begin{aligned} \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -1 & 2 \end{vmatrix} &= 1 \begin{vmatrix} 4 & -7 \\ -1 & 2 \end{vmatrix} - 4 \begin{vmatrix} 2 & -7 \\ 0 & 2 \end{vmatrix} + (-7) \begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} \\ &= 1(-2 - 4) - 4(4) + (-7)(-2) \\ &= -2 - 16 + 14 = 0. \end{aligned}$$

Problem

Given two non-parallel vectors \vec{u} and \vec{v} , can you find a ^{non-zero} vector \vec{w} that is orthogonal (i.e. perpendicular) to both \vec{u} and \vec{v} ? This is a vector that gives a non-zero volume for the parallelepiped generated by \vec{u}, \vec{v} and \vec{w} .

Idea: We will compute the determinant of \vec{u}, \vec{v} and a vector containing the vectors of the standard basis ($\vec{i}, \vec{j}, \vec{k}$), and the result will be "a scalar" with \vec{i}, \vec{j} and \vec{k} as coefficients (and is thus a vector).

Definition

If $\vec{u} = \langle a, b, c \rangle$ and $\vec{v} = \langle d, e, f \rangle$, the cross product of \vec{u} and \vec{v} is the vector

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix} = (bf - ec)\vec{i} - (af - dc)\vec{j} + (ae - bd)\vec{k} \\ &= \langle bf - ec, dc - af, ae - bd \rangle. \end{aligned}$$

Theorem

The vector $\vec{u} \times \vec{v}$ is perpendicular to both \vec{u} and \vec{v} .

Proof

Two vectors are perpendicular if and only if their dot product is 0:

$$\begin{aligned}\vec{u} \cdot (\vec{u} \times \vec{v}) &= \langle a, b, c \rangle \cdot \langle bf - ec, dc - af, ae - bd \rangle \\ &= \cancel{abf} - \cancel{aec} + \cancel{bdc} - \cancel{baf} + \cancel{cae} - \cancel{cbd} \\ &= 0.\end{aligned}$$

And the same is true for $\vec{v} \cdot (\vec{u} \times \vec{v})$. ◻

Example

Find a vector orthogonal to $\langle 1, 4, -7 \rangle$ and $\langle 2, -1, 4 \rangle$.

We compute the cross product.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & -7 \\ 2 & -1 & 4 \end{vmatrix} = \vec{i} \begin{vmatrix} 4 & -7 \\ -1 & 4 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & -7 \\ 2 & 4 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix}$$

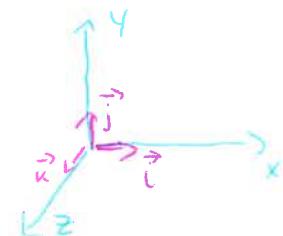
$$= \langle 9, -18, -9 \rangle$$

Orientation

The orientation for the dot product is given by the right hand rule: place your right hand and fold your fingers in the same orientation as the rotation from \vec{u} to \vec{v} (the shortest angle). Then your thumb indicates the orientation of $\vec{u} \times \vec{v}$.

Example

$$\vec{k} \times \vec{j} = -\vec{i} \quad \text{and} \quad \vec{j} \times \vec{k} = \vec{i}$$



Properties of the cross product.

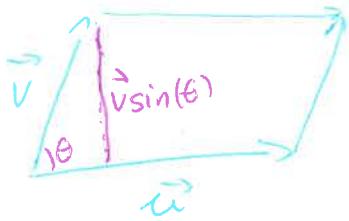
- (i) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- (ii) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
and
 $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- (iii) $|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin(\theta)$, where θ is the shortest angle from \vec{u} to \vec{v} .
- (iv) $|\vec{u} \times \vec{v}|$ is the area of the parallelogram generated by \vec{u} and \vec{v} .
- (v) \vec{u} and \vec{v} are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$.

Ideas for the proof

- (i) By the right-hand rule.
- (ii) It is a straightforward computation.
- (iii) Take $\vec{u} = \langle a, b, c \rangle$ and $\vec{v} = \langle d, e, f \rangle$. Then
$$\begin{aligned} |\vec{u} \times \vec{v}|^2 &= \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix} \right\|^2 \\ &= |(bf-ec, cd-af, ae-bd)|^2 \\ &= (bf-ec)^2 + (cd-af)^2 + (ae-bd)^2 \\ &= b^2f^2 + e^2c^2 + c^2d^2 + a^2f^2 + a^2e^2 + b^2d^2 - 2bcef - 2acd^2 - 2abde \\ &= (a^2 + b^2 + c^2)(d^2 + e^2 + f^2) - a^2d^2 - b^2e^2 - c^2f^2 - 2bcef - 2acd^2 - 2abde \\ &= (a^2 + b^2 + c^2)(d^2 + e^2 + f^2) - (ad + be + cf)^2 \\ &= |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2 \quad \text{by algebraic definition of dot product} \\ &= |\vec{u}|^2 |\vec{v}|^2 - |\vec{u}|^2 |\vec{v}|^2 \cos^2 \theta \quad \text{alternative definition of dot product} \\ &= |\vec{u}|^2 |\vec{v}|^2 (1 - \cos^2 \theta) \quad \text{put } |\vec{u}|^2 |\vec{v}|^2 \text{ in evidence} \\ &= |\vec{u}|^2 |\vec{v}|^2 \sin^2 \theta \quad \text{since } 1 = \sin^2 \theta + \cos^2 \theta. \end{aligned}$$

Then, $|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin(\theta)$ by removing the square.

(iv) $|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin(\theta)$ (by (iii)), so the area of the parallelogram below is $|\vec{u}| |\vec{v}| \sin(\theta)$.



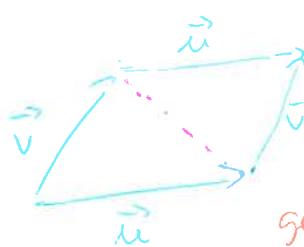
(v) If \vec{u} and \vec{v} are parallel, they generate a parallelogram of area 0.



Problems

(i) What is the area of the triangle with vertices $(1, 4, 6)$, $(-2, 5, -1)$ and $(1, -1, 1)$?

Solution: Find two vectors \vec{u} and \vec{v} from one vertex to another:



$$\vec{u} = (-2, 5, -1) - (1, 4, 6) = \langle -3, 1, -7 \rangle$$

$$\text{and } \vec{v} = (1, 4, 6) - (1, -1, 1) = \langle 0, 5, 5 \rangle.$$

Use property (iv), so the area of the parallelogram generated by \vec{u} and \vec{v} is $|\vec{u} \times \vec{v}|$:

$$\left| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -7 \\ 0 & 5 & 5 \end{vmatrix} \right| = \left| \vec{i} \begin{vmatrix} 1 & -7 \\ 0 & 5 \end{vmatrix} - \vec{j} \begin{vmatrix} -3 & -7 \\ 0 & 5 \end{vmatrix} + \vec{k} \begin{vmatrix} -3 & 1 \\ 0 & 5 \end{vmatrix} \right|$$

$$= \left| 40\vec{i} + 15\vec{j} - 15\vec{k} \right|$$

$$= \sqrt{40^2 + 15^2 + 15^2}$$

$$= 5\sqrt{8^2 + 3^2 + 3^2}$$

$$= 5\sqrt{82},$$

which is a little bit over 45 units².

(7)

- (iii) Find a vector that is perpendicular to the plane containing $(1, 4, 6)$, $(-2, -5, -1)$ and $(1, -1, 1)$.

This is the same plane as the one containing the triangle from the preceding example. Hence, the vector $\langle 40, 15, -15 \rangle$ is orthogonal to it.

References: James STEWART. Calculus, 8th edition, § 12.4.