

## Vector functions and space curves

10/18/2019

last class, we introduced vector-valued functions to express lines:  $\langle at + p_1, bt + p_2, ct + p_3 \rangle$ .

We can do the same with non-linear functions as components.

Definition

Given three real-valued functions  $f(t), g(t), h(t)$  (i.e. functions whose image is a real number),

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

is a vector-valued function.

How to think about this?

Think of  $t$  as time, and to the vector as the position vector of a particle at time  $t$ , where the  $x$ -coordinate is  $f(t)$ , ...

Example

$\vec{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$  is a vector-valued function and is defined for all values of  $t$  for which all three of

$$\begin{cases} t^3 \\ \ln(3-t) \\ \sqrt{t} \end{cases}$$

is defined. That is  $0 \leq t < 3$ , the domain of  $\vec{r}$ .

Continuity

The limit of a vector-valued function is given by the limits of the components: if  $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ ,

$$\lim_{t \rightarrow a} \vec{r}(t) = \left( \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right).$$

A vector-valued function  $\vec{r}$  is continuous at  $a$  if  $\lim_{t \rightarrow a} \vec{r}(t) = \vec{r}(a)$ .

(2)

Example

Find the limit as  $t \rightarrow 0$  of  $\vec{r}(t) = (1+t^3)\vec{i} + t e^{-t}\vec{j} + \frac{\sin(t)}{t}\vec{k}$ .

This is

$$\left( \lim_{t \rightarrow 0} (1+t^3) \right) \vec{i} + \left( \lim_{t \rightarrow 0} t e^{-t} \right) \vec{j} + \left( \lim_{t \rightarrow 0} \frac{\sin(t)}{t} \right) \vec{k}$$

$$= \vec{i} + \vec{k}.$$

Space curves

Suppose  $f(t)$ ,  $g(t)$  and  $h(t)$  are continuous function on an interval  $I$ . The set of images of  $\vec{r}(t) = (f(t), g(t), h(t))$  (seen as position vectors) as  $t$  varies throughout  $I$  is a space curve.

This is the set of points  $(f(t), g(t), h(t))$  when  $t \in I$ .

Space curves and motion

If we see a vector-valued function  $\vec{r}(t)$  as the position of a particle at time  $t$ , then a space curve is just the trajectory of the particle during an interval of time.

Example

Describe the curve defined by

$$\vec{r}(t) = \langle 1+t, 2+5t, -1+6t \rangle, \quad t \in \mathbb{R}$$

As seen last lecture, this is just the line of direction vector  $\langle 1, 5, 6 \rangle$  and passing through  $(1, 2, -1)$ .

This is also a curve.

③

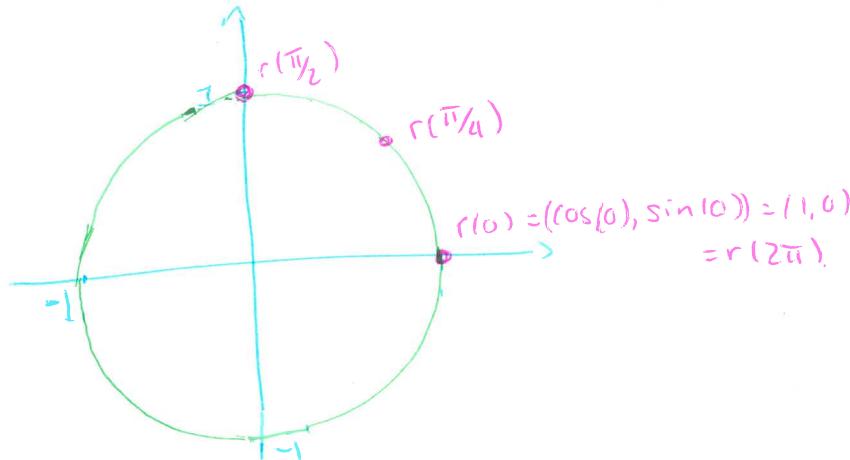
Example

Sketch the curve whose equation is

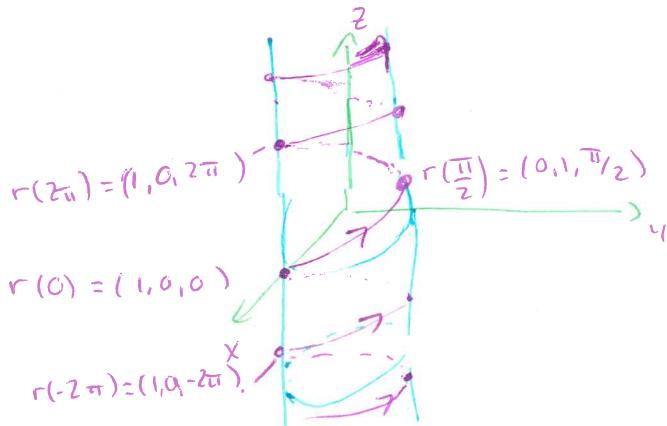
$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle.$$

(1) Look first at the projection on the  $xy$ -plane:  $\langle \cos(t), \sin(t) \rangle$ .

Drawing this curve for  $t \in [0, 2\pi]$ , I get



Hence the curve in three dimensions is at the surface of a cylinder of radius 1, centered at  $(0, 0, z)$ .



This curve is called the helix.

It is also possible to visualize it using Geogebra!

Example

What is the vector equation of the line segment from point  $P = (1, 3, -2)$  to  $Q = (2, 1, 3)$ ?

We already know how to compute the vector  $\vec{PQ}$  and give the equation of the line on which they lie.

But it could be simpler.

(4)

If we allow  $t$  to range from 0 to 1, we want to be in  $P$  at time  $t=0$ ,  $Q$  at time  $t=1$ , and gradually move from  $P$  to  $Q$ .

This can be done with the equation.

$$\vec{r}(t) = (1-t)\vec{P} + t\vec{Q}, \quad 0 \leq t \leq 1,$$

so we do not even have to compute the vector.

Plugging the values of  $\vec{P}$  and  $\vec{Q}$ , we get

$$\begin{aligned}\vec{r}(t) &= (1-t)\langle 1, 3, -2 \rangle + t\langle 2, -1, 3 \rangle \\ &= \langle 1-t+2t, 3-3t+t, -2+2t+3t \rangle \\ &= \langle 1+t, 3-4t, -2+5t \rangle, \quad 0 \leq t \leq 1.\end{aligned}$$

### Example

Can you find the curve for the intersection of the surfaces  $x^2+y^2=1$  and  $y+z=2$ .

(i) Identify what type of surfaces these are.

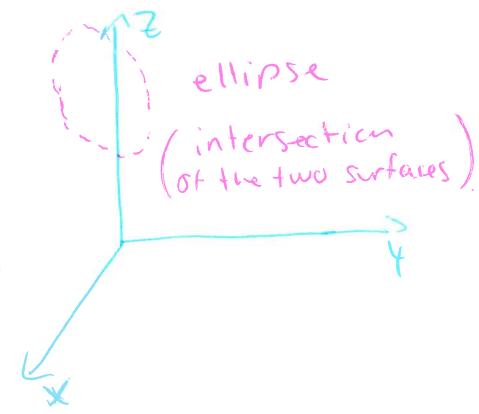
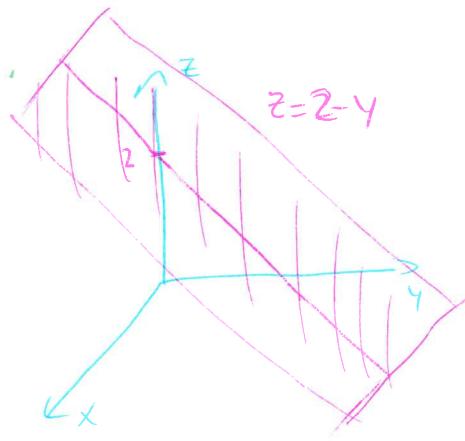
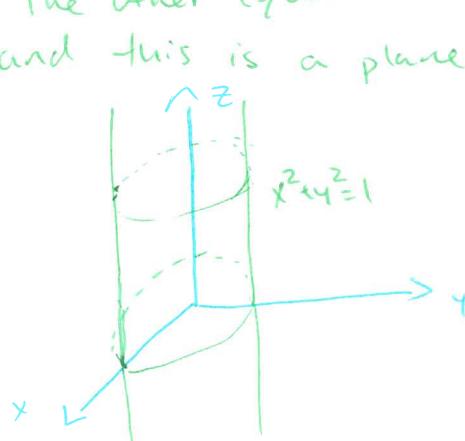
(ii) Sketch them, and sketch their intersection.

(iii) Compute the equation of the curve for the intersection

(i) As a 3D object, the one with equation  $x^2+y^2=1$  is a surface (because it has 2 (including  $z$ ) free variables). Moreover, for each value of  $t$ , the intersection of  $x^2+y^2=1$  with the plane  $z=t$  is a circle of radius 1 centered at  $(0,0,t)$ . So this surface is a cylinder.

The other equation also has two free variables (including  $x$ ), and this is a plane.

(ii)



(iii) The first equation, since we get a circle, suggests that the two first coordinates would be  $\cos(t)$  and  $\sin(t)$ , and that  $t$  would range from 0 to  $2\pi$ .

For the parametric equations, we have

$$x = \cos t, \quad y = \sin t.$$

What is  $z$ ? We know that  $z = 2 - y = 2 - \sin t$ .

Hence, the curve is defined by  $\vec{r}(t) = (\cos(t), \sin(t), 2 - \sin(t))$ .

### Example

What is the vector equation of the ellipse centered at  $(-3, 2)$  that has x-radius 5 and y-radius 1?

This is just a deformation of the circle centered at  $(-3, 2)$ . This circle has equation

$$\langle \cos t - 3, \sin t + 2 \rangle, \quad 0 \leq t \leq 2\pi.$$

However, this is a circle of radius 1. This is certainly correct for the y-radius here, but the x-radius should be 5, so we multiply the first coordinate by 5 ( $5\sin t$ ) to expand it:

$$\vec{r}(t) = \langle 5\cos(t) - 3, \sin(t) + 2 \rangle, \quad 0 \leq t \leq 2\pi$$

Reference: James STEWART. Calculus, 613.1.