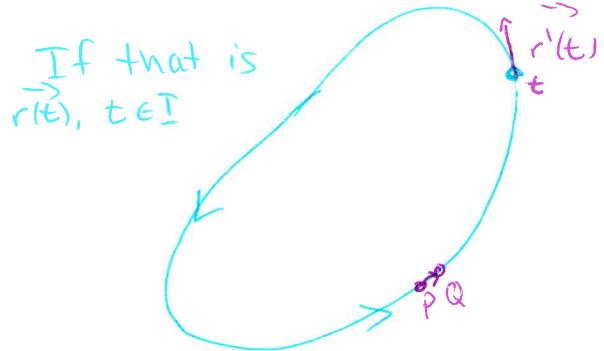


Last class, we introduced the vector valued functions as curves.

How can we find the tangent vector to a curve $\vec{r}(t)$, at point $(f(t), g(t), h(t))$ (for a given t)?



If that is $\vec{r}(t)$, $t \in I$

Assume you have two points

$$P = (f(t_1), g(t_1), h(t_1)) \text{ and}$$

$$Q = (f(t_1 + \Delta t), g(t_1 + \Delta t), h(t_1 + \Delta t))$$

on the curve, the instantaneous variation rate is given by the vector

$$\frac{\vec{Q} - \vec{P}}{\Delta t}, \text{ as } \Delta t \rightarrow 0. \text{ (Here, } \vec{P} \text{ and } \vec{Q} \text{ are}$$

the position vectors of P and Q , i.e. } $\vec{OP} = \vec{P}$ and $\vec{OQ} = \vec{Q}$ }

$$\lim_{\Delta t \rightarrow 0} \left\langle \frac{f(t_1 + \Delta t)}{\Delta t}, \frac{g(t_1 + \Delta t)}{\Delta t}, \frac{h(t_1 + \Delta t)}{\Delta t} \right\rangle$$

$$= \left\langle \lim_{\Delta t \rightarrow 0} \frac{f(t_1 + \Delta t)}{\Delta t}, \lim_{\Delta t \rightarrow 0} \frac{g(t_1 + \Delta t)}{\Delta t}, \lim_{\Delta t \rightarrow 0} \frac{h(t_1 + \Delta t)}{\Delta t} \right\rangle$$

$$= \langle f'(t_1), g'(t_1), h'(t_1) \rangle.$$

The derivative of the vector-valued function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\text{is } \vec{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}, \text{ if this limit exists.}$$

Equivalently, $\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$, provided the three functions f, g and h are differentiable in t .

The tangent vector of the curve at point P such that $\vec{P} = \vec{r}(t)$ is $\vec{r}'(t)$ if that vector exists and is non-zero.

The tangent line is the line passing by P with direction vector $\vec{r}'(t)$.

If $\vec{r}(t)$ is the displacement of an object, then $\vec{r}'(t)$ is its velocity vector, and $|\vec{r}'(t)|$ is its speed.

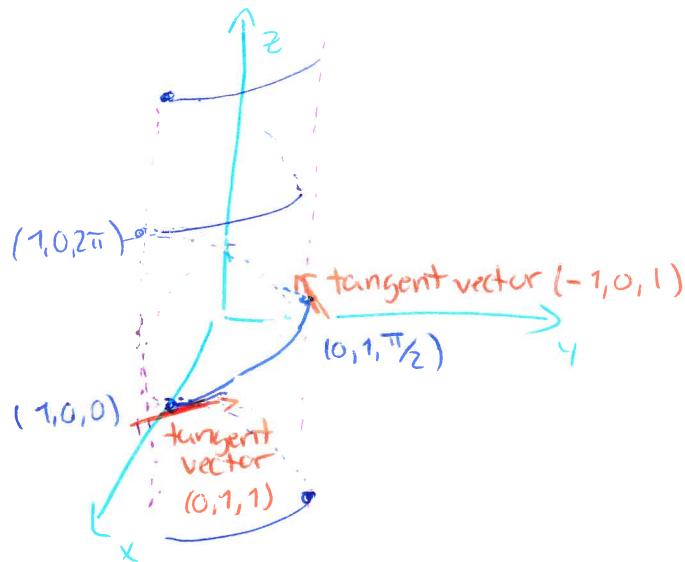
Example

Recall the equation of the helix:

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle, t \in \mathbb{R}.$$

Its derivative is

$$\vec{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle, t \in \mathbb{R}.$$



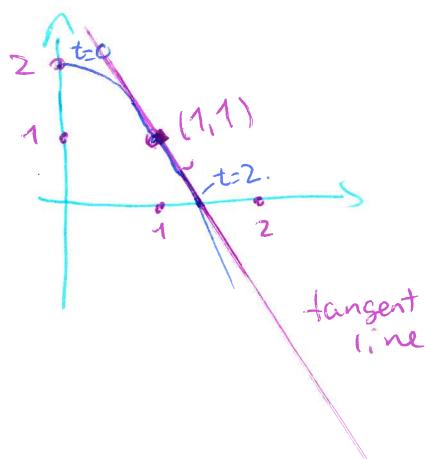
Can you sketch the tangent line passing... through $(1, 0, 2\pi)$?

Example

Sketch the 2D-curve $\vec{r}(t) = \sqrt{t}\vec{i} + (2-t)\vec{j}$ and the tangent line passing through $(1, 1)$. The domain of $\vec{r}(t)$ is $t \geq 0$.

(i) This curve is such that $x = \sqrt{t}$ and $y = 2 - t = 2 - x^2$.

So we can sketch $y = 2 - x^2$.



$$(i) \vec{r}'(t) = \frac{1}{2\sqrt{t}}\vec{i} - \vec{j}$$

So $\vec{r}'(1) = \frac{1}{2}\vec{i} - \vec{j}$ (which implies that $y = -2x + C$, a constant that can be determined by $1 = -2 \cdot 1 + C$. So $C = 3$, and $y = -2x + 3$).

is the direction vector of the tangent line passing through $(1, 1)$.

The second derivative of a vector function $\vec{r}(t)$ is $\vec{r}''(t)$, the derivative of $\vec{r}'(t)$.
 It represents the acceleration vector.

Differentiation rules

Let \vec{u} and \vec{v} be differentiable vector functions, f be a differentiable real-valued function and $c \in \mathbb{R}$. Then,

$$(i) \frac{d}{dt} [\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t) \quad \text{Addition.}$$

$$(ii) \frac{d}{dt} [c \cdot \vec{u}(t)] = c \cdot \vec{u}'(t) \quad \text{Scalar multiplication}$$

$$(iii) \frac{d}{dt} [f(t) \vec{u}(t)] = f'(t) \vec{u}(t) + f(t) \vec{u}'(t) \quad \text{Product rule}$$

$$(iv) \frac{d}{dt} [\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}(t) \cdot \vec{v}'(t) + \vec{u}'(t) \cdot \vec{v}(t) \quad \text{Dot product}$$

$$(v) \frac{d}{dt} [\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t) \quad \text{Cross product.}$$

$$(vi) \frac{d}{dt} [\vec{u}(f(t))] = \vec{u}'(f(t)) \cdot f'(t) \quad \text{Chain rule}$$

Example

Show that if $|\vec{r}(t)| = c$ (a constant), then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$ for all t . (You can think of a sphere, since it is an object that verify this).

$$\text{Since } |\vec{r}(t)| = c, \vec{r}(t) \cdot \vec{r}(t) = c^2.$$

$$\text{So, on the one hand, } \frac{d}{dt} [\vec{r}(t) \cdot \vec{r}(t)] = \frac{d}{dt} c^2 = 0.$$

$$\text{On the other hand, } \frac{d}{dt} [\vec{r}(t) \cdot \vec{r}(t)] = 2 \vec{r}(t) \cdot \vec{r}'(t). \quad (\text{rule iv}).$$

$$\text{Hence, } \vec{r}(t) \cdot \vec{r}'(t) = 0 \text{ for all } t, \text{ and } \vec{r}'(t) \text{ is always orthogonal to } \vec{r}(t).$$

It would be great to have something like the Fundamental Theorem of Calculus for vector-valued functions; sort of an inverse to the derivative.

Integrals

The integral of a vector-valued function is the integral of its components. If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\int \vec{r}(t) dt = \langle \int f(t) dt, \int g(t) dt, \int h(t) dt \rangle$$

To prove this, use the definition of an integral as the limit of a Riemann sum.

Example

Let $\vec{r}(t) = \langle -\sin(t), \cos(t), t \rangle$, $t \in \mathbb{R}$. (we saw earlier that this is the tangent vectors of the helix).

Then,

$$\begin{aligned} \int \vec{r}(t) dt &= \langle -\int \sin(t) dt, \int \cos(t) dt, \int t dt \rangle \\ &= \langle \cos(t), \sin(t), t \rangle + \langle c_1, c_2, c_3 \rangle, \quad c_i \in \mathbb{R}. \end{aligned}$$

Interpretation

Suppose we know the velocity of a function as t evolves, the integral of the velocity is the displacement vector.

Reference: James STEWART. Calculus, 8th edition. § 13.2.