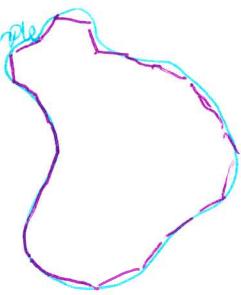


length of a curve

Let  $\vec{r}(t)$  be a curve defined by  $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ .

What is the length of the curve from  $t=a$  to  $t=b$ ?

2D-example

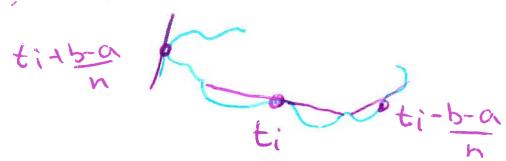


We can approximate the curves by line segments, and then compute their sum.

If we divide the curve into  $n$  segments, each one has length (in 2D)

$$\frac{b-a}{n} \cdot (\sqrt{f'(t_i)^2 + g'(t_i)^2}), \text{ where } t_i \text{ is in the } i\text{-th segment.}$$

Why?



Taking the limit of that sum, as  $n \rightarrow \infty$ , we get

$$\int_a^b \sqrt{f'(t)^2 + g'(t)^2} dt, \text{ to express the}$$

length of the curve between  $a$  and  $b$ .



In three dimensions, the length of the curve  $\vec{r}(t)$  from  $t=a$  to  $t=b$  is

$$\int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt$$

Example

The length of the helix  $\langle \cos(t), \sin(t), t \rangle$  between  $t=0$  and  $t=2\pi$  is

$$\int_0^{2\pi} \sqrt{(\sin^2(t) + \cos^2(t)) + 1} dt = \int_0^{2\pi} \sqrt{2} dt = (\sqrt{2}t) \Big|_0^{2\pi} = 2\sqrt{2}\pi.$$

## Definition

The curvature of a smooth curve (i.e. a differentiable curve with non-zero vector derivative) measures how quickly the curve changes direction at a point.

A curve is smooth if it has no sharp corners or cusps.

If the curve defined by  $\vec{r}(t)$ , the unit tangent vector at point  $t$  is

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}.$$

The curvature of  $\vec{r}(t)$  is computed by

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

Greek letter  $\kappa$  (κ)  
Kappa

## Example

A circle of radius  $a$  has curvature  $\frac{1}{a}$  everywhere:

$$\vec{r}(t) = \langle a\cos(t), a\sin(t) \rangle, \quad t \in [0, 2\pi].$$

$$\vec{r}'(t) = \langle -a\sin(t), a\cos(t) \rangle = a\vec{T}(t) \quad (\text{because } \vec{r}'(t) \text{ is a vector of length } a)$$

Hence, the curvature of the circle is

$$\kappa(t) = \frac{\|\langle -a\sin(t), a\cos(t) \rangle\|}{\|\langle a\cos(t), a\sin(t) \rangle\|} = \frac{1}{a}.$$

What does that mean? That the smaller a circle it is, the more curved it is locally.

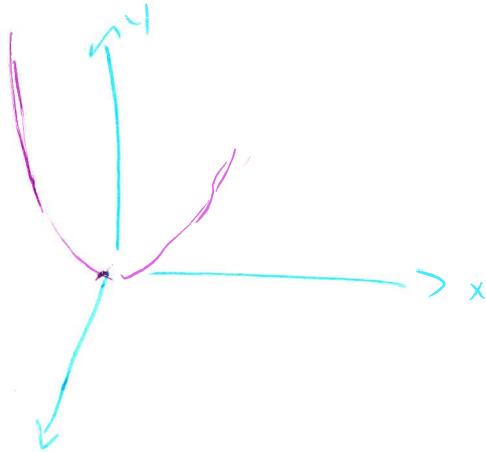
## Theorem

The curvature of the curve given by the function  $\vec{r}(t)$  is

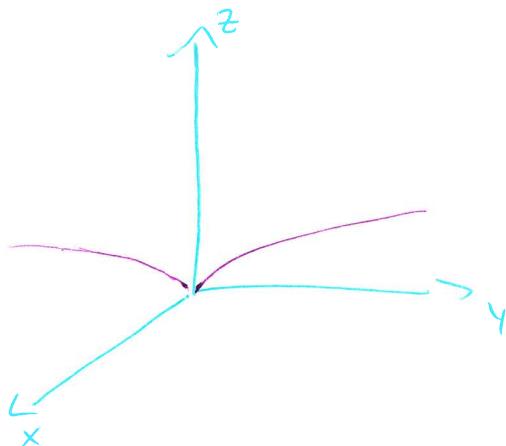
$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{\|\vec{r}'(t)\|^3}$$

Example

Find the curvature of the twisted curve  $\vec{r}(t) = \langle t, t^2, t^3 \rangle$  at a general position, and at  $t=0$ . Where should it be maximal?



but also



It is still smooth (think of a lace with a loose loop), but the curvature should be maximal close to  $t=0$ .

To compute it:

$$\vec{r}(t) = \langle 1, 2t, 3t^2 \rangle$$

$$\text{and } \vec{r}'(t) = \langle 0, 2, 6t \rangle.$$

$$\begin{aligned} \vec{r}'(t) \times \vec{r}''(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \vec{i} \begin{vmatrix} 2t & 3t^2 \\ 6t & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix} \\ &= 6t^2 \vec{i} - 6t \vec{j} + 2 \vec{k} \end{aligned}$$

The theorem says that

$$K(t) = \frac{|\langle 6t^2, -6t, 2 \rangle|}{\|\langle 1, 2t, 3t^2 \rangle\|^3} = \frac{2 \sqrt{9t^4 + 9t^2 + 1}}{\left(\sqrt{1 + 4t^2 + 3t^4}\right)^3}$$

At  $t=0$ , the curvature is 2.

Example

Find the curvature of the parabola  $y = x^2$  at  $(0,0)$ ,  $(1,1)$  and  $(2,4)$ . What should the limit as  $x \rightarrow \infty$  be?

This curve is defined by  $\vec{r}(t) = \langle t, t^2 \rangle$  for  $t \in \mathbb{R}$ .

Also,

$$\vec{r}'(t) = \langle 1, 2t \rangle = \vec{i} + 2t\vec{j}$$

$$\vec{r}''(t) = \langle 0, 2 \rangle = 2\vec{j}.$$

$$\text{Hence, } \vec{r}'(t) \times \vec{r}''(t) = (\vec{i} + 2t\vec{j}) \times 2\vec{j}$$

$$= \vec{i} \times 2\vec{j} + 2t\vec{j} \times 2\vec{j} \quad (\text{parallel vectors})$$

$$= 2\vec{k},$$

$$\text{and } |\vec{r}'(t) \times \vec{r}''(t)| = 2.$$

The length of  $\vec{r}'(t)$  is  $\sqrt{1+4t^2}$ , which gives the curvature:

$$K(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{2}{(1+4t)^{3/2}}.$$

At  $(0,0)$ , the curvature is 2.

At  $(1,1)$ , it is  $\frac{2}{\sqrt{5^3}} \approx 0.179$ .

At  $(2,4)$ , it is  $\frac{2}{(9)^{3/2}} = \frac{2}{27} \approx 0.074$ .

It seems to be decreasing...

$$\lim_{t \rightarrow \infty} K(t) = \lim_{t \rightarrow \infty} \frac{2}{(1+4t)^{3/2}} = 0.$$

That means that, as  $t \rightarrow \infty$ , the parabola resembles a straight line.

Reference: James STEWART. Calculus, 8<sup>th</sup> edition, § 13.3