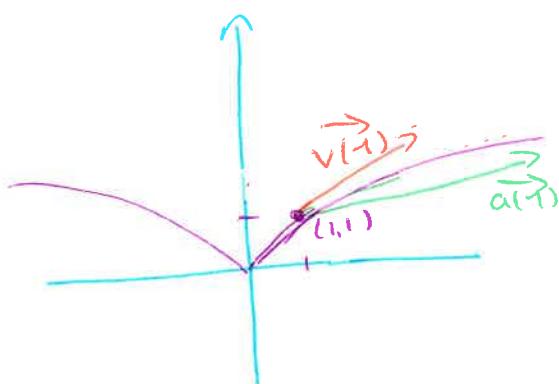


We introduced space curves as the trajectory of a particle over the time :- For a curve $\vec{r}(t)$:

- $\vec{r}(t)$ is the position vector at time t .
- $\vec{r}'(t)$ is the velocity vector at time t .
- $|r'(t)|$ is the speed at time t
- $\vec{r}''(t)$ is the acceleration at time t (it is a vector).
- $\int_a^t \vec{r}'(t) dt$ is the displacement vector.

Example

The position vector of an object moving in a plane is given by $\vec{r}(t) = t^3 \vec{i} + t^2 \vec{j}$. Find its velocity, speed, and acceleration when $t=1$, and illustrate it geometrically.



Fixing $x=t^3$ and $y=t^2$, we get $y=x^{2/3}$

Its velocity is

$$\vec{v}(t) = \vec{r}'(t) = \langle 3t^2, 2t \rangle,$$

Its speed is

$$|\vec{v}(t)| = \sqrt{9t^4 + 4t^2},$$

and its acceleration is

$$\vec{a}(t) = \vec{v}'(t) = \langle 6t, 2 \rangle$$

When $t=1$,

$$\vec{v}(1) = \langle 3, 2 \rangle, \quad \vec{a}(1) = \langle 6, 2 \rangle.$$

Example

A moving particle starts at an initial position $\vec{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\vec{v}(0) = \vec{i} - \vec{j} + \vec{k}$. Its acceleration is $\vec{a}(t) = 4t\vec{i} + 6t\vec{j} + \vec{k}$. Find its velocity and position at time t .

We know that

$$\vec{v}(t) = \int_0^t \vec{a}(t) dt + \vec{v}(0).$$

Hence,

$$\begin{aligned}\vec{v}(t) &= \int_0^t \langle 4t, 6t, 1 \rangle dt + \langle 1, -1, 1 \rangle \\ &= \langle 2t^2, 3t^2, t \rangle + \langle 1, -1, 1 \rangle.\end{aligned}$$

$$= \langle 2t^2+1, 3t^2-1, t+1 \rangle$$

is the velocity at time t .

In the same way,

$$\vec{r}(t) = \int_0^t \vec{v}(t) dt + \vec{r}(0),$$

and

$$\begin{aligned}\vec{r}(t) &= \int_0^t \langle 2t^2+1, 3t^2-1, t+1 \rangle dt + \langle 1, 0, 0 \rangle \\ &= \left\langle \frac{2t^3}{3} + t, \frac{t^3 - t}{2}, \frac{t^2}{2} + t \right\rangle + \langle 1, 0, 0 \rangle \\ &= \left\langle \frac{2t^3}{3} + t + 1, t^3 - t, \frac{t^2}{2} + t \right\rangle.\end{aligned}$$

is the position vector at time t .

In general,

$$\vec{v}(t) = \vec{v}(t_0) + \int_{t_0}^t \vec{a}(u) du$$

and

$$\vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^t \vec{v}(u) du.$$

Law of motion

We saw in the last unit Newton's second law of motion: $F = m \cdot a$. This of course has a vector version: $\vec{F} = m \vec{a}$.

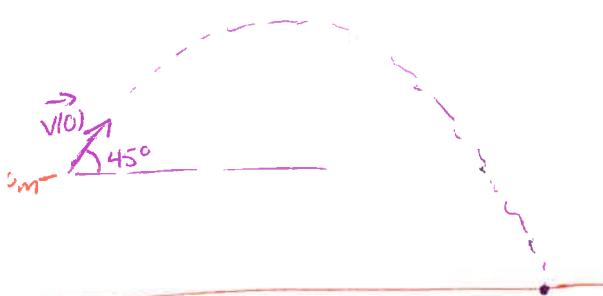
Projectile motion

When a projectile is fired, it has a speed and the direction is important. Also, there is a force acting on it: the gravity, whose magnitude is given by the second law of motion $|\vec{F}| = m \cdot \underbrace{9.8 \text{ m/s}^2}_{\text{Gravitational acceleration}}$. The direction of \vec{F} is the same as $-\vec{j}$.

Usually, we know what is the velocity and position at $t=0$.

Example

A projectile is fired with muzzle speed 150 m/s and angle of elevation 45° from a position 10 m above ground level. Where does the projectile hit the ground, and with what speed?



$$\text{Initial position: } \vec{r}(0) = \langle 0, 10 \rangle$$

$$\begin{aligned} \text{Initial velocity: } \vec{v}(0) &= \langle 150 \cdot \cos(45^\circ), 150 \cdot \sin(45^\circ) \rangle \\ &= \langle 75\sqrt{2}, 75\sqrt{2} \rangle \end{aligned}$$

$$\text{Acceleration vector: } \vec{a}(0) = \langle 0, -9.8 \rangle$$

$$\begin{aligned} \vec{v}(t) &= \vec{v}(0) + \int_0^t \langle 0, -9.8 \rangle du \\ &= \vec{v}(0) + \langle 0, -9.8t \rangle \\ &= \langle 75\sqrt{2}, 75\sqrt{2} - 9.8t \rangle. \end{aligned}$$

$$\begin{aligned} \text{Position vector: } \vec{r}(t) &= \vec{r}(0) + \int_0^t \langle 75\sqrt{2}, 75\sqrt{2} - 9.8u \rangle du \\ &= \langle 0, 10 \rangle + \langle 75\sqrt{2}t, 75\sqrt{2}t - 4.9t^2 \rangle \\ &= \langle 75\sqrt{2}t, 10 + 75\sqrt{2}t - 4.9t^2 \rangle, \end{aligned}$$

and the projectile touches the ground when the second component of $\vec{r}(t)$ is 0; this is at

$$t = \frac{-75\sqrt{2} \pm \sqrt{2 \cdot 75^2 + 196}}{-9.8} \approx 21.74 \rightarrow \text{Only one answer. Otherwise } t < 0, \text{ and it makes no sense.}$$

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And the distance to the muzzle is $(21.74) \cdot 75 \sqrt{2} \approx 2306$ meters

Components of acceleration

There are two parts to acceleration:

- The tangential acceleration is an acceleration in the direction of the motion (or of its tangent vector).
- The normal acceleration is an acceleration in the perpendicular direction. We should think of it as changing direction.

Definition

The unit normal vector of $\vec{r}(t)$ whose unit tangential vector is $\vec{T}(t)$ is given by

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

It is called normal, because $\vec{N}(t) \cdot \vec{T}(t) = 0$. To see that, notice that $\vec{N}(t) \cdot \vec{T}(t) = \frac{\vec{T}'(t) \cdot \vec{T}(t)}{\|\vec{T}'(t)\|} = \frac{1}{\|\vec{T}'(t)\|} (\vec{T}'(t) \cdot \vec{T}(t))$.

But we saw on Monday that if $\|\vec{T}'(t)\|$ is constant, then $\vec{T}'(t)$ is orthogonal to $\vec{T}(t)$, and $\vec{T}(t)$ is a unit vector.

To get the components of acceleration, we differentiate $\vec{v}(t)$ in two different ways:

• We know that $\vec{a}(t) = \frac{d}{dt} \vec{v}(t)$.

• Also, $\vec{v}(t) = \underbrace{\|\vec{v}(t)\|}_{\text{speed}} \cdot \vec{T}(t)$. Hence, $\vec{a}(t) = \|\vec{v}(t)\| \vec{T}'(t) + \|\vec{v}(t)\| \vec{T}(t) \cdot \vec{T}'(t)$.

$$= \|\vec{v}(t)\| \vec{T}'(t) + \|\vec{v}(t)\| \|\vec{T}'(t)\| \vec{N}(t)$$

$$\text{using } \vec{a} = \|\vec{v}(t)\| \vec{T}'(t) + \|\vec{v}(t)\|^2 \kappa \vec{N}(t)$$

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{v}(t)\|} = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

Theorem

The tangential acceleration component is given by

$$a_T(t) = |\vec{v}(t)|^1,$$

and the normal acceleration component by

$$a_N(t) = \kappa(t) |\vec{v}(t)|^2.$$

The acceleration is

$$\vec{a}(t) = a_T(t) \vec{T}(t) + a_N(t) \vec{N}(t)$$

Example

A particle moves with position function $\vec{r}(t) = \langle t^2, t^2, t^3 \rangle$.

Find the components of acceleration.

Tangential acceleration:

$$\vec{v}(t) = \langle 2t, 2t, 3t^2 \rangle, \text{ and } |\vec{v}(t)| = \sqrt{8t^2 + 9t^4}$$

$$\text{Also, } |\vec{v}(t)|^1 = \frac{16t+36t^3}{2\sqrt{8t^2+9t^4}} = \frac{8t+18t^3}{\sqrt{8t^2+9t^4}} = a_T(t).$$

Normal acceleration:

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

$$|\vec{r}'(t) \times \vec{r}''(t)| = \left| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 2t & 3t^2 \\ 2t & 2t & 6t \end{vmatrix} \right| = |\langle 6t^2, -6t^2, 0 \rangle| = 6\sqrt{2}t^2,$$

so

$$a_N(t) = \frac{6\sqrt{2}t^2}{|\vec{r}'(t)|} = \frac{6\sqrt{2}t^2}{\sqrt{8t^2+9t^4}}$$

Reference: James STEWART, Calculus, 8th edition. §13.4.