

Functions of several variables: level curves

In the past weeks, we looked at surfaces and defined their equations, that have two independent variables.

We now look at them as functions.

Definition

A function f of two variables is a rule that assigns to each ordered pair of real numbers (x,y) a unique real number $f(x,y)$, for (x,y) in a set D (the domain). The set of values $\{f(x,y) \mid (x,y) \in D\}$

Example

The function $z = f(x,y) = 3x - 2y$ is the function that represents the plane $3x - 2y - z = 0$, which is the plane with normal vector $\langle 3, -2, -1 \rangle$ passing through the origin.

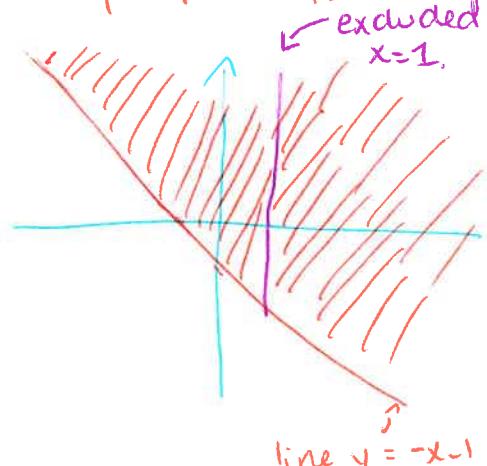
Its domain is \mathbb{R}^2 and its range is \mathbb{R} .

Example

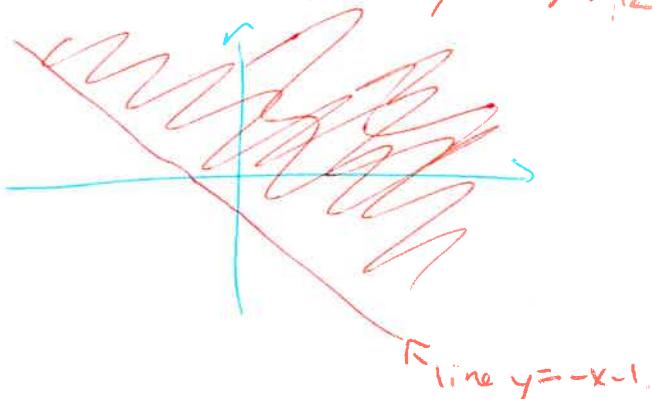
Find the domain and range of the following functions, and sketch the domain.

$$\text{a) } f(x,y) = \frac{\sqrt{x+y+1}}{x-1}, \quad \text{b) } f(x,y) = \sqrt{x+y+1}, \quad \text{c) } f(x,y) = x \ln(y^2-x).$$

$$\text{a) } D = \{(x,y) \mid x+y \geq -1 \text{ and } x \neq 1\}, \text{ and the range is } \mathbb{R}$$



$$\text{b) } D = \{(x,y) \mid x+y \geq -1\}, \text{ range: } z \geq 0$$

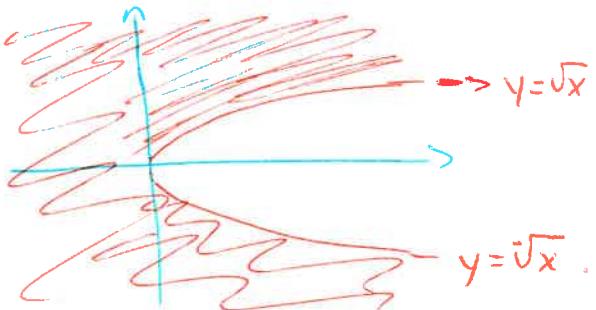


(2)

$$\begin{aligned}
 \text{Q) } D &= \{(x,y) \mid y^2 - x \geq 0\} \\
 &= \{(x,y) \mid y^2 \geq x\} \\
 &= \{(x,y) \mid y \geq \sqrt{x} \text{ or } y \leq -\sqrt{x}\}
 \end{aligned}$$

The range is \mathbb{R} .

To see it, look at all the values of $f(x,y)$ when $x=1$. The range of $f(y) = \ln(y^2 - 1)$ is \mathbb{R} .



Example

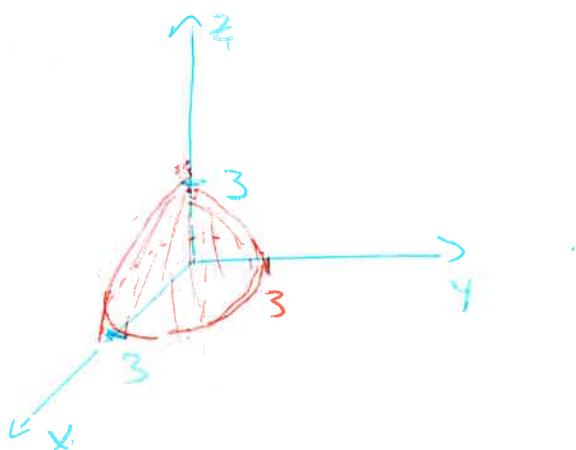
Find the domain and range of $g(x,y) = \sqrt{9-x^2-y^2}$.

Can you sketch the graph of $z = \sqrt{9-x^2-y^2}$.

$D = \{(x,y) \mid x^2 + y^2 \leq 9\}$, which is the disk of radius 3 centered at the origin.

The range is $0 \leq z \leq 3$.

You can rewrite $z = \sqrt{9-x^2-y^2}$ as $z^2 + x^2 + y^2 = 9$, with the constraint that $z \geq 0$. Hence, it is a half-sphere of radius 3 centered at origin that lies above the xy-plane.



Level curves

The level curves of a function f of two variables are the curves with equation $f(x,y) = k$, where k is a constant in the range of f .

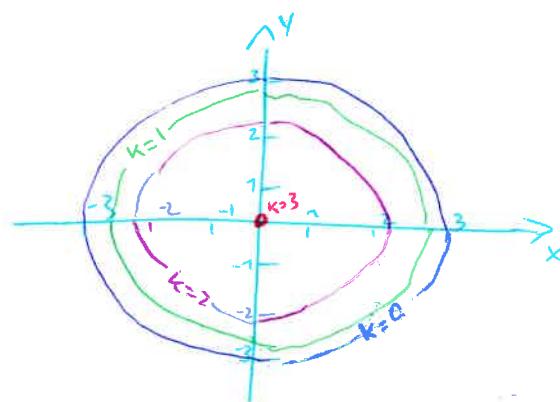
We project them to the xy -plane.

Example

- The level curves on a topographical map.

Example

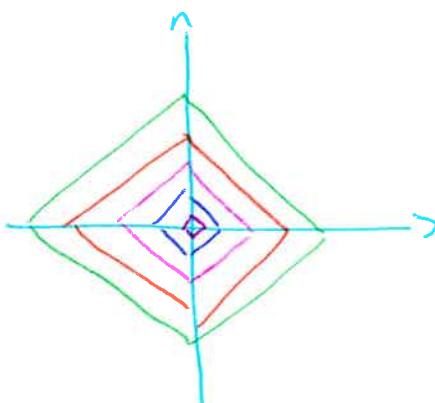
Sketch the curves for $k \in \{0, 1, 2, 3\}$ for $f(x,y) = \sqrt{9-x^2-y^2}$.



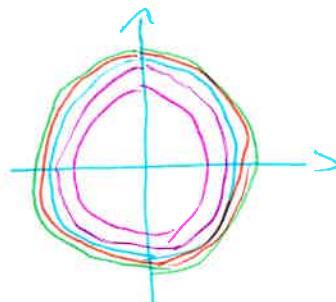
$$\begin{aligned}
 k=0 \quad & 9 = x^2 + y^2 \\
 k=1 \quad & 8 = x^2 + y^2 \\
 k=2 \quad & 5 = x^2 + y^2 \\
 k=3 \quad & 3 = \sqrt{9 - x^2 - y^2} \\
 & \Rightarrow x = y = 0
 \end{aligned}$$

Example

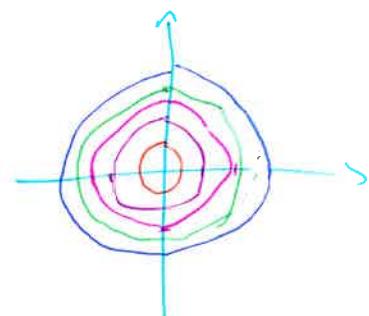
Describe the functions that can be associated with the level curves.



(a pyramid)



(a paraboloid,
 $x^2 + y^2 = z$)

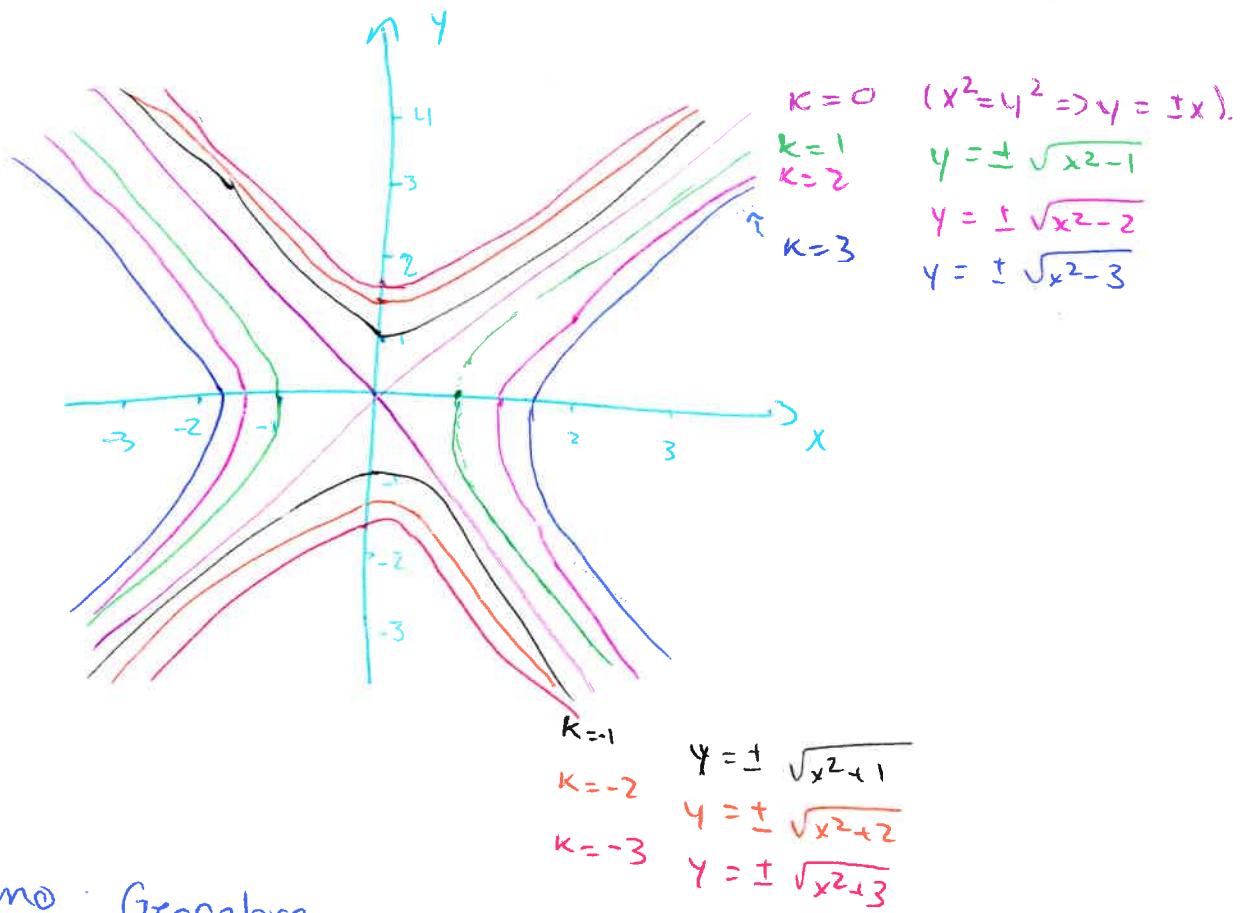


(a cone
 $x^2 + y^2 = z^2$)

Note that some functions have disconnected components for their level curves. (4)

Example

Draw the level curves of $x^2 - y^2 = f(x,y)$ for $k \in \{-3, -2, -1, 0, 1, 2, 3\}$.



Software demo : Geogebra.

Reference : James STEWART. Calculus, 8th edition, § 14.1