

Math 8- Lecture 1

- Ice breaker, go-around-the-table: 1st name, expected major (if any),
10 min what place feels home to you?

- Syllabus: some highlights:

10 min

- assignments
- Honor Principle, how it applies
- tutorials office hours, x-har
- Policies
- Question?



One more policy: no more calculator.

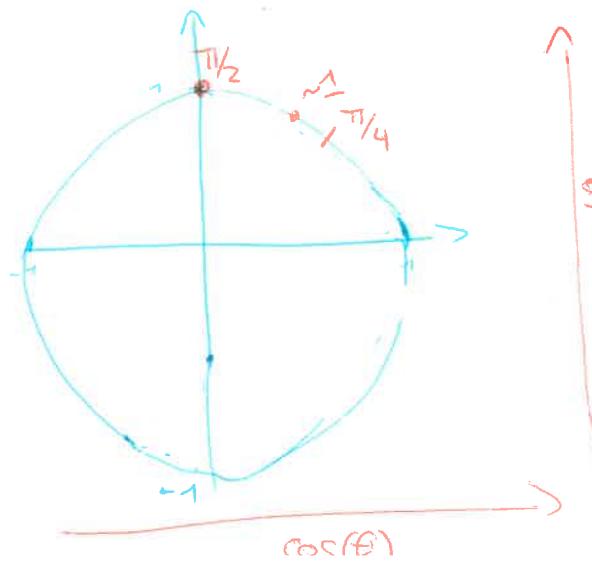
- A first exercise in that setting: compute $\sin(1)$.

- What does $\sin(1)$ mean?
◦ What is 1? Angle, radians.

- What we know about it:

- $-1 \leq \sin(x) \leq 1$ for $x \in \mathbb{R}$
notation: x a real number.

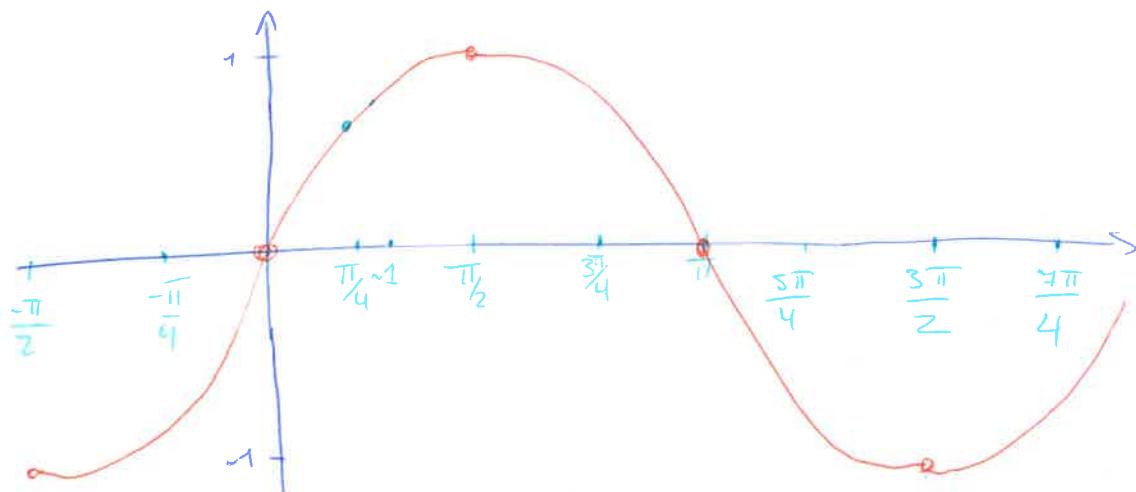
- $\frac{\pi}{4} \leq 1 \leq \frac{\pi}{2}$ and, with the picture, we know that



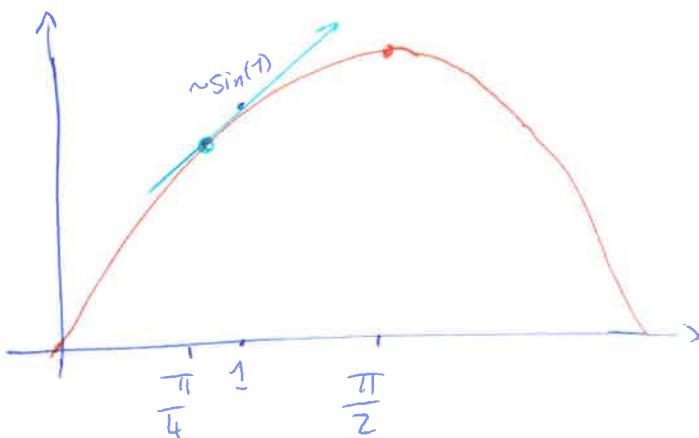
$$\sin\left(\frac{\pi}{4}\right) \leq \sin(1) \leq \sin\left(\frac{\pi}{2}\right)$$

$$\sin(6) \Rightarrow \frac{\sqrt{2}}{2} \leq \sin(1) \leq 1$$

The other thing we have an idea of is the graph of $\sin(x)$:



To get an approximation of $\sin(1)$, we can start from $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, and draw the tangent line close to it



Then, a closer approximation of $\sin(1)$ would be

$$\sin(1) \approx \sin\left(\frac{\pi}{4}\right) + \sin'\left(\frac{\pi}{4}\right) \cdot \left(1 - \frac{\pi}{4}\right)$$

$$= \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) \cdot \left(1 - \frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{2} \left(1 + 1 - \frac{\pi}{4}\right) \approx 0.707, 1.21 \approx 0.8591$$

(Exact answer is close to 0.8415)

(3)

On the graph, we see that the point on the straight line (the one we computed) seems to be above the actual value of $\sin(1)$.

This means, we should "bend down" our approximation of $\sin(x)$ to get a better estimation of $\sin(1)$.

1-2-all Question: How can we do it?

1-1

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Recap An outline of the method:

We need two things:

- a nice (i.e. smooth) function f that is hard to compute.
- a real number a (in the domain of f), around which we want to get an estimation of f

We will approximate f by a function g that is easy to compute (i.e. a polynomial).

For g to be appropriate, one must have,

$$f(a) = g(a),$$

and, ideally,

$$f'(a) = g'(a),$$

$$f''(a) = g''(a),$$

:

$$f^{(n)}(a) = g^{(n)}(a).$$

(4)

We then say that g approximates f in the neighborhood of a .

Definition

The n -th degree Taylor polynomial of f centered at a , denoted $T_n(x)$, is the unique polynomial of degree n such that

$$f(a) = T_n(a)$$

$$f'(a) = T_n'(a)$$

⋮ ⋮

$$f^{(n)}(a) = T_n^{(n)}(a).$$

Question: How do we find what T is like?

4x4
5

Reminder: a polynomial is a function of the form

$$P(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n.$$

If they are lost after 3 minutes:

find the problem with

$$T_2(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2.$$

Definition (continued)

$$T_n(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!}.$$

A MacLaurin polynomial is a Taylor polynomial centered at $a=0$.

2x2
3 Exercise: Verify that $T_n(x)$ and $f(x)$ have the same n -th derivative.

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Solution

This page will skip if we don't have time. (5)

The terms of degree smaller than n will be canceled by the n -th derivative. (If that sounds hard, come to the x-har on Tuesday).

So we only need to verify that the n -th derivatives of f and $\frac{f^{(n)}(a)(x-a)^n}{n!}$ are equal when evaluated in a .

Moreover,

$$\left(\frac{f^{(n)}(a)(x-a)^n}{n!} \right)' = \frac{f^{(n)}(a)}{n!} \left((x-a)^n \right)'$$

Why?
The variable
is x .

Finally,

$$\left((x-a)^n \right)^{(n)} = \left(n(x-a)^{n-1} \right)^{(n-1)}$$

$$= \left(n \cdot (n-1) \cdot (x-a)^{n-2} \right)^{(n-2)}$$

= ...

$$= \left(n \cdot (n-1) \cdot (n-2) \cdots (n-i) \cdot (x-a)^{n-i-1} \right)^{(n-i-1)}$$

= ...

$$= \underbrace{\left(n \cdot (n-1) \cdot (n-2) \cdots (n-n+1) \right)}_{n!} \cdot (x-a)^{n-n}$$

$$= n!$$

Thus, $\left(\frac{f^{(n)}(a)(x-a)^n}{n!} \right)^{(n)} = \frac{f^{(n)}(a)}{n!} \cdot n! = f^{(n)}(a)$, the n -th derivative of f . ■

Final question:

What about $\sin(1)$ from the beginning?

We approximated $\sin(x)$ around $a = \frac{\pi}{4}$.

$$(\sin(x))' = \cos(x), \quad \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$(\sin(x))'' = -\sin(x), \quad -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$(\sin(x))''' = -\cos(x), \quad -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

With Geogebra, plot

$$f = \sin(x)$$

$$T_0(x) = \frac{\sqrt{2}}{2}$$

$$T_1(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \left(x - \frac{\pi}{4}\right)$$

$$T_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4} \left(x - \frac{\pi}{4}\right)^2$$

$$T_3(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4} \left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12} \left(x - \frac{\pi}{4}\right)^3$$

and

$$f(a) \approx 0.841$$

$$T_0(a) \approx 0.707$$

$$T_1(a) \approx 0.859$$

$$T_2(a) \approx 0.843$$

$$T_3(a) \approx 0.841$$