

Problem

What is the limit?

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2-y^2}{x^2+y^2} ?$$

If that limit exists, then it should be equal to

$$\lim_{y \rightarrow 0} \frac{-y^2}{y^2} \quad \text{and to} \quad \lim_{x \rightarrow 0} \frac{x^2}{y^2}$$

But these are respectively -1 and 1. So the limit is not defined
 (picture on Geogebra) - Give intuition.

Definition

Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b) . Then, we say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is L ,

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$$

if for every number $\epsilon > 0$, there is a corresponding number $\delta > 0$ such that if $(x,y) \in D$ and $\underbrace{\sqrt{(x-a)^2 + (y-b)^2}}_{(x,y) \text{ within distance } \delta \text{ from } (a,b)} < \delta$, then $|f(x,y) - L| < \epsilon$.

(Comparison with one-dimensional functions)

Example

If $f(x,y) = \frac{xy}{x^2+y^2}$, does $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ exist?

Two ways of approaching $(0,0)$ are on the lines $x=y$ and $x=0$.

(i) If $x=y$,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2} = \lim_{x \rightarrow 0} \frac{x^2}{2x^2} = \frac{1}{2}.$$

(ii) If $x=0$,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2} = \lim_{y \rightarrow 0} \frac{0 \cdot y}{0^2+y^2} = \lim_{y \rightarrow 0} \frac{0}{y^2} = 0.$$

Since the two paths do not lead to the same answer, the limit does not exist.

Example

Does $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^4}{x^2+y^8}$ exist?

Choosing to approach it from $y=0$, we get

$$\lim_{y \rightarrow 0} \frac{0 \cdot y^4}{0^2+y^8} = \lim_{y \rightarrow 0} \frac{0}{y^8} = 0.$$

Choosing $y=0$ or $x=y$ would lead to the same.

However, choosing $x=y^4$, we get

$$\lim_{y \rightarrow 0} \frac{y^8}{y^8+y^8} = \frac{1}{2}.$$

The limit does not exist.

Example

Does $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2+y^2}$ exist?

Choosing $y=0$,

$$\lim_{x \rightarrow 0} \frac{3x^2 \cdot 0}{x^2+0^2} = \lim_{x \rightarrow 0} \frac{3 \cdot 0}{x^2} = 0.$$

Choosing $x=0, x=y$ or $x^2=y$, we get 0 as the limit.

Try to prove this is true:

The limit exists if, for every $\epsilon > 0$, there exists δ such that

$$\sqrt{x^2+y^2} < \delta \Rightarrow \left| \frac{3x^2y}{x^2+y^2} \right| < \epsilon.$$

Since $\frac{x^2}{x^2+y^2} \leq 1$,

$$\left| \frac{3x^2y}{x^2+y^2} \right| \leq 3|y| \leq 3 \cdot \sqrt{x^2+y^2} < 3\delta.$$

Therefore, setting $\delta = \frac{\epsilon}{3}$, we get

$$\left| \frac{3x^2y}{x^2+y^2} \right| < 3\delta = \epsilon.$$

So the limit exists and is equal to 0.

Continuity

A function f of two variables is called continuous at (a,b) if

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b).$$

We say f is continuous on D if f is continuous at every point (a,b) in D .

Example

- A polynomial is always continuous: $x + x^2y$ is continuous everywhere on \mathbb{R}^2 .

* $g(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$ is not continuous in $(0,0)$, since

$$\lim_{(x,y) \rightarrow (0,0)} g(x,y) \text{ does not exist.}$$

* $f(x,y) = \begin{cases} \frac{3xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is continuous in $(0,0)$

- Where is $\arctan(x/y)$ continuous?

- $\arctan(x)$ is continuous everywhere
- x/y is not defined in $y=0$, and continuous everywhere.
- So $\arctan(x/y)$ is continuous everywhere, except where $y=0$

Extra problems

(i) Prove that $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2+y^2}}$ exists.

(ii) Compute $\lim_{(x,y) \rightarrow (3,2)} e^{\sqrt{2x-4}}$. Where is $e^{\sqrt{x-y}}$ continuous?

(iii) Does $\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$ exist?

Reference: James STEWART. Calculus, 8th edition. §14.2.