

What is the meaning of the derivative, geometrically?

- For real-valued functions of one variable: it is the slope of the function. It also gives the tangent line that approximates the function close to a point.
- For vector-valued functions: it is the tangent vector to that function, and it gives the direction of the line tangent to the curve.
- For surfaces? There are many slopes...
 - In some cases, it gives a tangent plane (if that exists).
 - We can use it to approximate the value of a function.

Example

Let $f(x,y) = x^2 + yz$ be a paraboloid.

- Does f admit a tangent plane in $(1,2,5)$?

We will see later that it is the case, since its equation is a polynomial.

- What is the slope of f in $(1,2,5)$?

- The slope of $g(x) = x^2 + 4$ is $2x$, which in $x=1$ is worth 2.

So the line $\{z = 2x + 3, y = 2\}$ is tangent to the paraboloid at $(1,2,5)$.

- Equivalently, with $h(y) = y^2 + 1$ in $y=2$, the line $\{z = 4y - 3, x = 1\}$ is tangent to f in $(1,2,5)$.

Definition

Let f be a function of two variables and P be a plane that is not vertical, so that $f(a,b) = P(a,b) = c$, for some $a,b,c \in \mathbb{R}$.

We say that P is tangent of f in (a,b) if

$$\lim_{(x,y) \rightarrow (a,b)} \frac{f(x,y) - P(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0.$$

Meaning: The numerator is the distance between f and P (in the z -coordinate) and the denominator is the distance between (x,y) and (a,b) (where f and P are equal).

As we get closer to (a,b) (in any direction), we want that the distance between f and P be very small.

Example

Check that the plane containing the lines $\{2x+3=z, y=2\}$ and $\{z=4y-3, x=1\}$ is tangent to $f(x,y)=x^2+y^2$ in $(1,2,5)$.

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The first line can be written as $\langle t, 2, 2t+3 \rangle = \vec{r}_1(t)$ and the second one as $\langle 1, t, 4t-3 \rangle = \vec{r}_2(t)$. To find the plane containing both, we use the cross product:

$$\langle 1, 0, 2 \rangle \times \langle 0, 1, 4 \rangle = \langle -2, -4, 1 \rangle,$$

so the plane has equation $-2x - 4y + z = -5$, or $z = 2x + 4y - 5$. It is tangent to f if the following limit is 0:

$$\begin{aligned} \lim_{(x,y) \rightarrow (1,2)} \frac{x^2+y^2 - (2x+4y-5)}{\sqrt{(x-1)^2 + (y-2)^2}} &= \lim_{(x,y) \rightarrow (1,2)} \frac{(x^2-2x+1) + (y^2-4y+4)}{\sqrt{(x-1)^2 + (y-2)^2}} \\ &= \lim_{(x,y) \rightarrow (1,2)} \frac{\sqrt{(x-1)^2 + (y-2)^2}}{\sqrt{(x-1)^2 + (y-2)^2}} \\ &= 0. \end{aligned}$$

Then, this is the plane tangent to f .

See on Geogebra.

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Proposition

If there exists a tangent plane to f , it contains the lines tangent to f on a given plane

Example

The function $f(x,y) = \frac{2xy}{\sqrt{x^2+y^2}}$ does not admit a tangent plane at $(0,0)$.

- Setting $x=0$, the function $\therefore g(y) = \frac{0}{\sqrt{y^2}} = 0$ leads to $(0, y, 0)$ being in the values of the function
- Setting $y=0$, the function $\therefore h(x) = 0$ gives $(x, 0, 0)$ being in the function. (x-axis)
- Since $x=0$ and $y=0$ are part of the surface, the tangent plane, if it exists, should be $z=0$.
- However, on the line $x=y$, $f(x,y) = \frac{2x^2}{\sqrt{2x^2}} = \sqrt{2}|x|$, and we cannot define a line tangent to the function $\sqrt{2}|x|$ close to 0.

So there is no tangent plane

Geogebra.

Definition

The function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable at (s,t) if its graph has a non-vertical tangent plane at the point $(s,t, f(s,t))$. If the plane is $P(x,y) = ax+by+c = z$, then the derivative of f at (s,t) is $f'(s,t) = \langle a, b \rangle$.

It is also called the directional derivative.

Application

Given a differentiable function f in (a,b) , we can approximate the values of f whenever (x,y) is close enough to (a,b) :

$$f(x,y) \approx f(a,b) + f'(a,b) \cdot \begin{pmatrix} x-a \\ y-b \end{pmatrix}$$

dot
product

Example

Let $f(x,y) = x^2 + y^2$. (We computed the tangent plane on p-2.)

Then, $f(1.1, 1.9) \approx 5 + 2 \cdot (0.1) + 4(-0.1) \approx 4.8$.

by taking $a=1, b=2$

Checking with a calculator, we get $1.1^2 + 1.9^2 = 4.82$

References: Marcial's handouts (see Section 2)

- James STEWART. Calculus, 8th edition, §14.4