Math 8 - Lecture 22
Partial derivatives

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Last class, we introduced the derivative of a function of two variables through tangent planes. Let us now look at the point of view of partial derivatives.

Definition

The partial derivative of $f(x,y)$ with respect to x at (a,b), denoted $f_x(a,b)$ is $f_x(a,b) = g'(a)$, where $g(x) = f(x,b)$ (b fixed).

Similarly, $f_y(a,b) = h'(b)$, where $h(y) = f(a,y)$.

Since g and h are defined as function of one variable, we can define their derivative in the usual way (with differentiation's rules, or with the definition with the limit).

Notation

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = f_x$$

and $$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = f_y$$

Rule for finding partial derivatives of $f(x,y)$

(i) To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.

(ii) To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

Example

If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.

Solution

$$f_x(2,1) = (3x^2 + 2xy^3)_{x=2,y=1} = 12 + 4 = 16$$

$$f_y(2,1) = (3x^2y^2 - 4y^2)_{x=2,y=1} = 12 - 4 = 8.$$

Example

If $f(x,y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(i) $\frac{\partial f}{\partial x} = \frac{1}{1+y}\cos\left(\frac{x}{1+y}\right)$ with the chain rule

(ii) $\frac{\partial f}{\partial y} = \frac{-x}{(1+y)^2}\cos\left(\frac{x}{1+y}\right)$

Implicit definition

In one dimension, a function $y = f(x)$ is defined implicitly if it is not defined by an equation with y isolated (on one side of the equation).

Example

The relation $x^2 + y^2 = 25$ is the implicit definition of the two functions $y = \pm\sqrt{25 - x^2}$

Implicit functions can be differentiated using the chain rule:

$$\frac{\partial}{\partial x}(x^2 + y^2) = \frac{\partial}{\partial x} 25$$

$$\Rightarrow \quad 2x + 2y\frac{\partial y}{\partial x} = 0$$

$$\Rightarrow \frac{\partial y}{\partial x} = \frac{-2x}{2y} = -\frac{x}{y}.$$

The same thing can be done for functions of two variables.

Example

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $e^z = xyz$.

(i) $\frac{\partial z}{\partial x} e^z = \frac{\partial z}{\partial x} xyz = y \frac{\partial z}{\partial x} xz \overset{\text{product rule}}{=} y(xz' + z)$

On the other hand,

$\frac{\partial z}{\partial x} e^z = e^z \cdot \frac{\partial z}{\partial x} = e^z z'$ Here, remember z is a function of x.

Thus, solving for z', we get

$$\frac{\partial z}{\partial x} = z' = \frac{yz}{e^z - xy}.$$

(ii) $\frac{\partial z}{\partial y} e^z = e^z \frac{\partial z}{\partial y}$

$\frac{\partial z}{\partial y} xyz = x \frac{\partial z}{\partial y}(yz) = x\left(z + y\frac{\partial z}{\partial y}\right)$

Hence,

$$\frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}$$

Higher derivatives (second-order, third derivative, etc.)

If f is a function of two variables, then so are f_x and f_y, so we can define second partial derivatives: $f_{xy}, f_{xx}, f_{yx}, f_{yy}$.

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2} \qquad f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \qquad f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2}$$

! ∂^2 does not mean the square of the derivative.

Example

Find all the second partial derivatives of $f(x,y) = x^3 + x^2y^3 - 2y^2$.

On page 2, we already found that

$$f_x(x,y) = 3x^2 + 2xy^3 \quad \text{and} \quad f_y(x,y) = 3x^2y^2 - 4y$$

Therefore,

$$f_{xx}(x,y) = 6x + 2y^3 \qquad f_{yx}(x,y) = 6xy^2$$

$$f_{xy}(x,y) = 6xy^2 \qquad f_{yy}(x,y) = 6x^2y - 4.$$

Notice that $f_{xy} = f_{yx}$. This is not a coincidence...

Theorem (Clairaut, Young, Euler, Schwartz...)

Let f be a function of x and y.

If f_{xy} and f_{yx} are continuous near (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Example

Let $f(x,y) = \ln(x+2y)$.

$$f_{xy} = \frac{\partial}{\partial x}\frac{2}{x+2y} = \frac{-2}{(x+2y)^2} \quad \text{and} \quad f_{yx} = \frac{\partial}{\partial y}\frac{1}{x+2y} = \frac{-2}{(x+2y)^2},$$

and indeed $f_{xy} = f_{yx}$.

Example

If $f(x,y) = x^3y^2 + \arcsin(x)$, find f_{xy}.

Using $f_{yx} = f_{xy}$, we get

$$f_{yx} = \frac{\partial}{\partial x} 2x^3y = 6x^2y = f_{xy}$$

Reference: James STEWART. Calculus, 8th edition. § 14.3.