

## Tangent planes.

Last Friday, we introduced tangent planes geometrically. We see them today as obtained from the partial derivatives.

Remember from last Friday the following: "If a tangent plane exists, then it contains the tangent lines".

More specifically, the partial derivatives give us some tangent lines.

Theorem

If the partial derivatives  $f_x$  and  $f_y$  exist near  $(a,b)$  and are continuous at  $(a,b)$ , then  $f$  is differentiable at  $(a,b)$ .

This implies that there is a plane that is tangent to  $f$  in  $(a,b)$ .

Proposition

Suppose  $f$  has continuous partial derivatives. An equation of the tangent plane to the surface  $z=f(x,y)$  at the point  $(a,b, f(a,b))$  is

$$z = \underline{f(a,b)} + \underline{f_x(a,b)}(x-\underline{a}) + \underline{f_y(a,b)}(y-\underline{b}).$$

$\underline{\quad}$  = constant terms

Remarks

- This is the equation of a plane.  
Note that it is not possible to have exponents for  $x, y$  and  $z$ .
- The point  $(a,b, f(a,b))$  is in that plane.

(2)

Example

We spent a lot of time Friday proving that  $f(x,y) = x^2 + y^2$  has tangent plane  $z = 2x + 4y - 5$ , in  $(1,2,5)$

The partial derivatives of  $f$  are  $f_x = 2x$  and  $f_y = 2y$ .

Hence,  $f_x(1,2) = 2$  and  $f_y(1,2) = 4$ .

They are continuous, so the plane is

$$z = 5 + 2(x-1) + 4(y-2)$$

$$= 5 + 2x - 2 + 4y - 8$$

$$= 2x + 4y - 5.$$

Linear approximations

Last Friday, we saw we can approximate a differentiable function  $f$  near  $(a,b)$

$$f(x,y) \approx f(a,b) + f'(a,b) \cdot \langle x-a, y-b \rangle$$

<sup>†</sup>  
directional  
derivative.

$$= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

(Intuition)

That approximation is a linear approximation or the tangent plane approximation.

Remark: It is much easier now than before, since before we had to find a tangent plane.

(3)

## Example

Show that  $f(x,y) = xe^{xy}$  is differentiable at  $(1,0)$  and approximate  $f(1.1, -0.1)$

We have.  $f_x(x,y) = e^{xy} + xye^{xy}$ , that is continuous  
 $f_x(1,0) = 1$ .

and  $f_y(x,y) = x^2e^{xy}$  (also continuous) and  
 $f_y(1,0) = 1$ .

So, near  $(1,0)$ ,

$$\begin{aligned} f(x,y) &\approx f(1,0) + 1 \cdot (x-1) + 1 \cdot (y-0) \\ &= 1 + (x-1) + y \\ &= x+y. \end{aligned}$$

In particular,

$$f(1.1, -0.1) \approx 1.$$

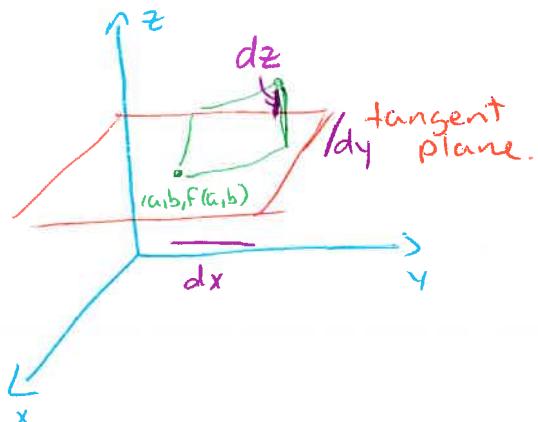
(The actual value of  $f(1.1, -0.1) \approx 0.985$ )

## Differentials

The differentials  $dx$  and  $dy$  represent small movements in the direction of the  $x$ - and  $y$ - axis respectively.

For functions of two variables,  $dx$  and  $dy$  are independent. The total differential,  $dz$ , is defined by

$$dz = f_x(x,y)dx + f_y(x,y)dy.$$



Example

- If  $z = f(x,y) = x^2 + 3xy - y^2$ , find  $dz$ .
- Compare it with  $\Delta z$  when  $x$  changes from 2 to 2.05 and  $y$  from 3 to 2.96. (Note that  $f(2.05, 2.96) = 13.6449$ .)

$$\begin{aligned} dz &= f_x(x,y) dx + f_y(x,y) dy \\ &= (2x+3y)dx + (3x-2y)dy \end{aligned}$$

- When  $x$  changes from 2 to 2.05,  $dx = 0.05$ .
- When  $y$  changes from 3 to 2.96,  $dy = -0.04$ .

Then,  $dz = (2(2)+3(3)) \cdot 0.05 + (3 \cdot 2 - 2 \cdot 3)(-0.04)$   
 $= 0.65$

$$\begin{aligned} \text{Also, } \Delta z &= 13.6449 - f(2,3) \\ &= 13.6449 - (4 + 18 - 9) \\ &= 13.6449 - 13 \\ &= 0.6449. \end{aligned}$$

Notice that  $\Delta z \approx dz$ , but  $dz$  is much easier to compute.

Example

Find a linear approximation at  $(0,0)$  of  $e^x \cos(xy) = f(x,y)$

Also, find the differential and difference with  $f(0.1, -0.1)$ , provided that  $f(0.1, -0.1) \approx 1.1051$ .

The linear approximation is

$$f(x,y) \approx e^0 \cos(0) + 1 \cdot (x-0) = 1+x.$$

because

$$f_x(x,y) = e^x \cos(xy) - e^y \sin(xy) \text{ and } f_y(x,y) = -x e^y \sin(xy).$$

The differential is then  $dz = dx$ , so when  $x$  moves from 0 to 0.1, this is  $dz = 0.1$ .

$$\text{Also, } \Delta z = 1.1051 - 1 = 0.1051 \text{ and } dz \approx \Delta z.$$