

## Chain Rule

Theorem (Chain rule, version 1)

Suppose that  $z = f(x, y)$  is a differentiable function and  $x$  and  $y$  are themselves functions of  $t$ :  $x = g(t)$  and  $y = h(t)$ , and they are both differentiable.

Then,  $z$  is differentiable in  $t$ , and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

↑ partial derivative  
 derivative of a function  
 "d" of one variable  
 "d"

Example

If  $z = x^2y + 3xy^4$ , where  $x = \sin 2t$  and  $y = \cos t$ , find  $\frac{dz}{dt}$  when  $t=0$ .

With the chain rule, we find

$$\frac{\partial f}{\partial x} = 2xy + 3y^4 \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 + 12xy^3.$$

Also,

$$\frac{dx}{dt} = 2\cos(2t) \quad \text{and} \quad \frac{dy}{dt} = -\sin t.$$

Hence,

$$\frac{dz}{dt} = 2(xy + 3y^4)\cos(2t) - (x^2 + 12xy^3)\sin t$$

For example, at  $t=0$ ,  $x=0$  and  $y=1$ , which gives

$$\frac{dz}{dt} = 6\cos(0) - 0\sin(0) = 6$$

Notice that

- (i) We can leave  $x$  and  $y$  in the equation, as they were part of the question.
- (ii) It is much easier to use the chain rule than to substitute first and then compute the derivative with the product rule.

### Example

The pressure (in kPa), the temperature (in K) and the volume (in L) of a mole of an ideal gas are related through the equation

$$P = 8.31 \frac{T}{V}$$

Find the rate at which the pressure is changing (in kPa/s) when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is increasing at a rate of 0.2 L/s, starting at 100 L.

Using the chain rule,

$$\begin{aligned} \frac{dP}{dt} &= \frac{\partial P}{\partial T} \frac{dT}{dt} + \frac{\partial P}{\partial V} \frac{dV}{dt} \\ &= \frac{8.31}{V} \cdot 0.1 + \frac{-8.31 T \cdot 0.2}{V^2} \\ &= \frac{8.31}{100} \left( 0.1 - \frac{300 \cdot 0.2}{100} \right) \\ &= \frac{8.31}{100} \cdot -0.5 \\ &= -0.04155 \end{aligned}$$

So the pressure is decreasing at a rate of  $-0.04155 \text{ kPa/s}$ .

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## Theorem (Chain rule, version 2)

Suppose that  $z = f(x, y)$ ,  $x = g(s, t)$  and  $y = h(s, t)$  are differentiable.

Then,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} .$$



partial derivatives.

## Example

If  $z = (x-y)^5$ ,  $x = s^2t$  and  $y = st^2$ , find  $\frac{\partial z}{\partial s}$  and  $\frac{\partial z}{\partial t}$ .

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \\ &= 5(x-y)^4 \cdot 2st + -5(x-y)^4 t^2 \\ &= 5t(x-y)^4 (2s-t)\end{aligned}$$

$$\begin{aligned}\text{and } \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \\ &= 5(x-y)^4 \cdot s^2 - 5(x-y)^4 \cdot 2st \\ &= 5s(x-y)^4 (s-2t).\end{aligned}$$

## Polar coordinates

You already know the cartesian  $(x, y, z)$  coordinates system, but some functions are easier to explain in polar coordinates. These are for two dimensions.

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Main idea: To replace  $x$  and  $y$  (i.e. movements along perpendicular axes), we move in terms of angle and distance.

$r$ : distance from the origin

$\theta$ : angle between the positive  $x$ -axis

Every point  $(x, y)$  can be rewritten as  $(r \cos \theta, r \sin \theta)$ .

Conversely, we can find  $(r, \theta)$  by writing  $r = \sqrt{x^2 + y^2}$ ,  $\theta = \tan^{-1}(\frac{y}{x})$

Here,  $r$  and  $\theta$  are functions of the variables  $x$  and  $y$ .

### Example

If  $x$  increases of 0.3 m starting at 2 meters, and  $y$  decreases from -1.5 m from 8m, what is the variation in  $r$ ? in  $\theta$ ?

with the chain rule,

$$\frac{dr}{dt} = \frac{\partial r}{\partial x} \frac{dx}{dt} + \frac{\partial r}{\partial y} \frac{dy}{dt}$$

That implies that

$$\begin{aligned}\frac{dr}{dt} &= \left( \frac{2x}{\sqrt{x^2+y^2}} \right)_{\substack{x=2 \\ y=8}} 0.3 + \left( \frac{-y}{\sqrt{x^2+y^2}} \right)_{\substack{x=2 \\ y=8}} (-1.5) \\ &= \frac{1}{\sqrt{4+64}} (0.6 - 12) \\ &= -\frac{11.4}{\sqrt{68}}\end{aligned}$$

So the variation in  $r$  is  $\frac{-11.4}{\sqrt{68}} \approx -1.382$  m.

In e,

$$\begin{aligned}
 \frac{d\theta}{dt} &= \frac{\partial\theta}{\partial x} \frac{dx}{dt} + \frac{\partial\theta}{\partial y} \frac{dy}{dt} \\
 &= \left( \frac{y}{(1+y^2)x^2} \right) \Big|_{x=2, y=8} \cdot 0.3 + \left( \frac{1}{x(1+y^2)} \right) \Big|_{x=2, y=8} (-1.5) \\
 &= \frac{-4}{x^2 y^2} \Big|_{x=2, y=8} \cdot 0.3 + \frac{1}{x^2 y^2} \Big|_{x=2, y=8} (-1.5) \\
 &= \frac{-8}{4+64} \cdot 0.3 + \frac{1}{2+32} (-1.5) \\
 &= \frac{1}{34} (-4 \cdot 0.3 - 1.5) \\
 &= -\frac{2.7}{34}
 \end{aligned}$$

So the variation in  $\theta$  is  $-\frac{2.7}{34} \approx -0.079$  radians

Reference: James STEWART. calculus, 8<sup>th</sup> edition. § 14.5