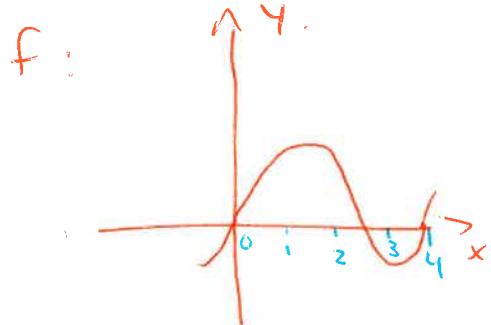
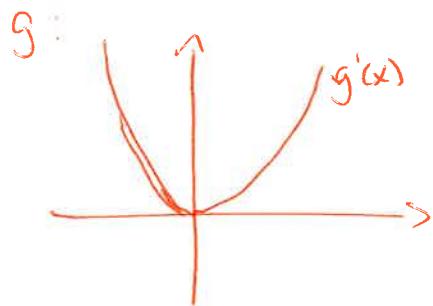


Example Let the graph below be the derivative of $f(x)$. Where is f maximal? minimal?



The derivative has zeroes in 0, 2.5 and 4. They are the only places where minima or maxima can appear. Here, 0 and 4 are local minima, and 2.5 is a maximum. (that can be seen by taking the integral).



Where is $g(x)$ maximal? minimal?

The only candidate is in 0, since this is the only zero of the derivative. However, if $g'(x)$ looks like x^2 , so $g(x)$ must resemble $\frac{x^3}{3}$, that has no minimum, nor maximum.

Summary

For functions of one variable,

- If f has a minimum or a maximum in a , then $f'(a)=0$.
- It happens that $f'(a)=0$ and that a is not a minimum nor a maximum.

How can we generalize that to functions of two variables?

Definition

A function of two variables has a local maximum (respectively minimum) at (a,b) if $f(x,y) \leq f(a,b)$ (resp. $f(x,y) \geq f(a,b)$) for all (x,y) near (a,b) . The number $f(a,b)$ is a local maximum (resp. minimum) value.

If $f(x,y) \leq f(a,b)$ for all (x,y) in the domain of f , then (a,b) is a global maximum.

Theorem

If f has a local maximum or minimum at (a,b) and the first-order partial derivatives of f exist, then $f_x(a,b)=0$ and $f_y(a,b)=0$.

A point (a,b) such that $f_x(a,b)=f_y(a,b)=0$ is a critical point. (It might not give rise to a minimum or a maximum).

Example

Find the minima and maxima of $f(x,y) = x^3 - 3x + 3xy^2$.

The partial derivatives are

$$f_x(x,y) = 3x^2 - 3 + 3y^2 \quad \text{and} \quad f_y(x,y) = 6xy.$$

They are equal to zero when:

- $f_x(x,y) = 0 \Leftrightarrow x^2 + y^2 = 1$ and $f_y(x,y) = 0 \Leftrightarrow x=0$ or $y=0$.
(i.e. on the circle of radius 1)
Centered at the origin

Hence, the critical points are

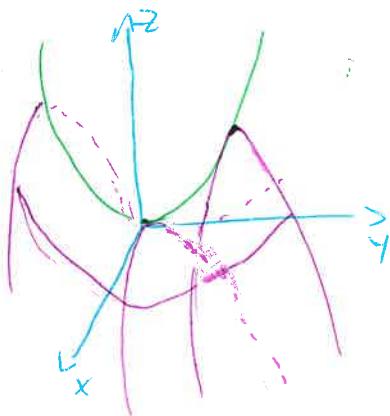
(a,b)	$f(a,b)$	min./max/other
$(0,1)$	0	nothing?
$(0,-1)$	0	nothing?
$(1,0)$	-2	minimum?
$(-1,0)$	2	maximum?

We lack information to tell whether or not it is a extremum. ③

Example

Find all the extrema of $f(x,y) = x^2 - y^2$.

The only candidate is when $2x=2y=0$, hence the origin.



However, if we look at the intersection with the yz -plane, the origin seems to be a maximum.

On the xz -plane, the origin seems to be a minimum.

Such a point is called a saddle point.

The following is a way to distinguish critical points.

Second Derivative test

Suppose the second partial derivatives of f are continuous near (a,b) , and suppose (a,b) is a critical point (i.e. $f_x(a,b)=f_y(a,b)=0$). Let

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

- (a) If $D > 0$ and $f_{xx}(a,b) > 0$, then (a,b) is a local minimum.
- (b) If $D > 0$ and $f_{xx}(a,b) < 0$, then (a,b) is a local maximum.
- (c) If $D < 0$, then (a,b) is not a minimum nor a maximum.

Notice that if $D=0$, one cannot say anything with this test.

Example

The origin for $f(x,y) = x^2 - y^2$ fits in case (c).

$$f_{xx}(x,y) = 2, f_{yy}(x,y) = -2, f_{xy}(x,y) = 0$$

Example

For $f(x,y) = x^3 - 3y + 3xy^2$ (continued from page 2),

$$f_{xx}(x,y) = 6x, \quad f_{yy}(x,y) = 6x, \quad f_{xy}(x,y) = 6y.$$

Hence,

(a,b)	D(a,b)	min/max/other
(0,1)	-36	nothing
(0,-1)	-36	nothing
(1,0)	36	minimum
(-1,0)	36	maximum.

Geogebra?

Applications

This can be used to solve optimisation problems.

Example

The extrema of what function are you looking for if you want to have the shortest distance from the point (-2,1,0) to the plane $x+2y+z=4$?

The distance from (-2,1,0) to (x,y,z) is

$$d = \sqrt{(x+2)^2 + (y-1)^2 + z^2}.$$

On the plane $x+2y+z=4$, $z = 4 - x - 2y$. Hence, the distance from (-2,1,0) to any point in that plane is

$$d = \sqrt{(x+2)^2 + (y-1)^2 + (4-x-2y)^2}$$

It will be minimal whenever its square will be (since \sqrt{x} is a monotonic.)

To find the minimum of d^2 , we calculate its partial derivatives:

$$d^2 = (x+2)^2 + (y-1)^2 + (4-x-2y)^2$$

$$\frac{\partial^2}{\partial x} (x,y) = 2(x+2) - 2(4-x-2y) \text{ and } \frac{\partial^2}{\partial y} (x,y) = 2(y-1) + 4(4-x-2y).$$

Solving for $\frac{\partial^2}{\partial x} (x,y) = \frac{\partial^2}{\partial y} (x,y) = 0$, we get

$$4x + 4y - 4 = 10y + 4x - 18 = 0$$

And this happens only when $x = -\frac{4}{3}$, $y = \frac{7}{3}$. (and hence, $z = \frac{2}{3}$)

Because of the nature of the function, this is the minimum, and the minimum distance is $\sqrt{(-\frac{4}{3}+2)^2 + (\frac{7}{3}-1)^2 + (\frac{2}{3})^2} = \frac{\sqrt{24}}{3} = \frac{2\sqrt{6}}{3}$.

We can also find the extrema on a bounded domain, D.

If we want to know the minimum and maximum of a function over D:

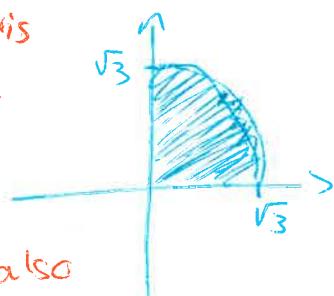
- (i) Find the value of f at its critical points in D.
- (ii) Find the extreme values on the boundary of D.
- (iii) The largest of these (from steps (i) and (ii)) is the absolute maximum value over D, and the smallest is the absolute minimum value.

Example

Find the extrema of $f(x,y) = xy^2$ over $D = \{(x,y) | x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}$

- (i) The critical points satisfy $y^2 = 2xy = 0$, and this is the whole x-axis. Since f is nonnegative over D and 0 on the x-axis, f has a minimum on the x-axis.

- (ii) On the y-axis, the function is also 0 and thus also a minimum.



- On $x^2+xy^2=3$ (the other part of the boundary), the function (6)
 xy^2 can be rewritten as $x(3-x^2) =: g(x)$.

$g(x)$ has critical points when $3-3x=0$, so in $x=1$. Since $g''(x)<0$,
this is a maximum, and $f(1, \sqrt{2}) = 2$ is the absolute maximum
of f over D . The minimum is 0 over the x - and y -axes.

Reference: James STEWART. Calculus, 8th edition. §14.7.