

## Sequences and series.

## I - Sequences

Question: What is the pattern of this sequence?

$$1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \dots$$

The pattern is  $\frac{1}{n!} = \prod_{i=1}^n \frac{1}{i}$ , for  $n \in \{0, 1, 2, \dots\}$ .  
 $\approx \left( \prod_{i=1}^n \frac{1}{i} = \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{n} \cdot \frac{1}{n} \right)$ .

Definition

A sequence is a function  $f: \mathbb{N} \rightarrow \mathbb{R}$ .

We write it  $\{f(n)\}_{n \in \mathbb{N}} = \{f(n)\}_{n=0}^\infty$ .

Example

$\left\{ \frac{1}{n!} \right\}_{n \in \mathbb{N}}$  is the sequence whose first terms are above.

Definition

The limit of the sequence, if it exists, is a real number  $L$  such that, for all  $\epsilon > 0$ , there exists a number  $N$  such that

$$|a_n - L| < \epsilon, \quad \forall \underset{n \rightarrow \infty}{n} \geq N$$

for all.

Then, we say  $\{a_n\}_{n \in \mathbb{N}}$  converges to  $L$ .

Question: Which of these have limit? (2)

a)  $\left\{ \frac{1}{n!} \right\}_{n \in \mathbb{N}}$

Try them!

b)  $\left\{ (-1)^n \right\}_{n \in \mathbb{N}}$

2½ minutes, by yourself

c)  $\left\{ (-1)^n / n \right\}_{n \in \mathbb{N}}$

It is okay to have  
only hints

d)  $\left\{ \sin(nx) \right\}_{n \in \mathbb{N}}, \text{ for some } x \neq 0, \text{ fixed}$

e)  $\left\{ \sin\left(\frac{x}{n}\right) \right\}_{n \in \mathbb{N}}, \text{ for some } x \neq 0, \text{ fixed}$

f)  $\{n\}_{n \in \mathbb{N}}$

g)  $\left\{ \frac{n!}{n^n} \right\}_{n \in \mathbb{N}}$ .

## Solutions

a), c), e) and g) converge to 0

b), d) and f) diverge. (Remember? The limit must be a real number!)

Some arguments. (tell them all in class, write for c) and e))

a) For all  $n \geq 1$ ,  $0 \leq \frac{1}{n!} \leq \frac{1}{n}$ , and  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .

By squeezing,  $\lim_{n \rightarrow \infty} \frac{1}{n!} = 0$ .

b)  $(-1)^n$  alternates according to the parity of  $n$ , so the difference between two consecutive terms is always 2.

c) For every  $\epsilon > 0$ , we set  $N = \frac{1}{\epsilon}$ , and for all  $n > N$ ,

$$\left| \frac{(-1)^n}{n} - 0 \right| = \frac{1}{n} < \frac{1}{N} = \epsilon.$$

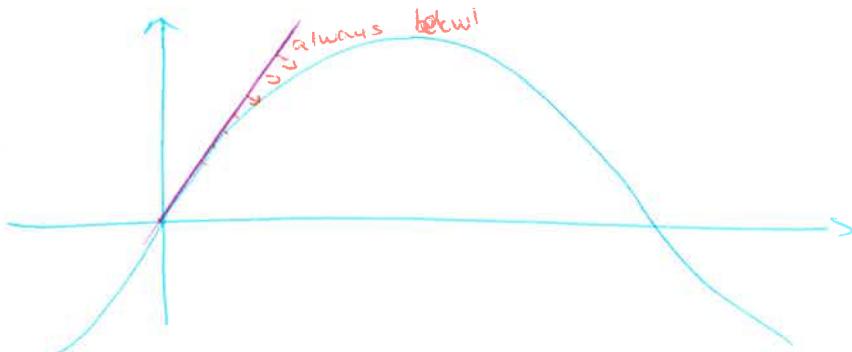
limit

(3)

d) For example, if  $x = \frac{\pi}{2}$ , then  $\sin(nx)$  takes values  $-1, 0, 1, 0,$  and repeats. This never converges.

e) Claim: when  $0 \leq x \leq \frac{\pi}{2}$ ,  $\sin(x) \leq x$

To convince yourself, draw the tangent of  $\sin(x)$  at  $x=0$ . In the range  $[0, \frac{\pi}{2}]$ , this slope is maximal, and the tangent line has equation  $y=x$ . Since the function is smooth, it is always below it.



This means that  $0 \leq \sin\left(\frac{x}{n}\right) \leq \frac{x}{n}$ .

Since  $\lim_{n \rightarrow \infty} \frac{x}{n} = 0$ ,  $\lim_{n \rightarrow \infty} \sin\left(\frac{x}{n}\right) = 0$ , by squeezing.

f) Infinity is not a real number!

$$g) \frac{n!}{n^n} = \underbrace{\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \frac{4}{n} \cdots \frac{n-1}{n}}_{<1} \cdot \overbrace{\frac{n}{n}}^{=1}$$

This product is very small!

Moreover,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n!}{n^n} &= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \underbrace{\left( \frac{2}{n} \cdot \frac{3}{n} \cdot \frac{4}{n} \cdots \frac{n-1}{n} \cdot \frac{n}{n} \right)}_{<1} \\ &< \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \end{aligned}$$

Since  $\frac{n!}{n^n}$  is always positive, it converges to 0.

## General Technique

- Get an idea of what the limit should be
- Prove it with the definition.

3 min: Notes comparison - Find a partner, exchange your notes, and comment on the notes of your partner.  
It is important to have good note taking skills!

Same limit laws for the sequences.

Let  $\{a_n\}_{n \in \mathbb{N}}$  and  $\{b_n\}_{n \in \mathbb{N}}$  be two convergent sequences

- $\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$
- $\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n$
- $\lim_{n \rightarrow \infty} c a_n = c \lim_{n \rightarrow \infty} a_n$
- $\lim_{n \rightarrow \infty} a_n b_n = (\lim_{n \rightarrow \infty} a_n)(\lim_{n \rightarrow \infty} b_n)$
- $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}$ , if  $\lim_{n \rightarrow \infty} b_n \neq 0$
- $\lim_{n \rightarrow \infty} (a_n)^p = \left(\lim_{n \rightarrow \infty} a_n\right)^p$ , if  $p > 0$ , and  $a_n > 0$ .
- If  $\{c_n\}_{n \in \mathbb{N}}$  is such that  $a_n \leq c_n \leq b_n$ , for all  $n > N$ , and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = L$ , then  $\lim_{n \rightarrow \infty} c_n = L$  (Squeeze theorem).

Many more criteria in the book, or in Marcia's notes (section 2)

Question: For what values of  $r$  does  $(r^n)_{n \in \mathbb{N}}$  converge?

Make some examples, and come with a conjecture. 1 min.

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Solution:

The sequence  $\{r^n\}_{n \in \mathbb{N}}$  is convergent if  $-1 < r \leq 1$ , and divergent otherwise. Moreover,

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Question: What about  $\lim_{n \rightarrow \infty} \sum_{k=0}^n r^k$ ?

remember  
this is  $r^0 + r^1 + \dots + r^n$

In quadruples

4 min

- For what values of  $r$  are you sure it diverges?
- For what values of  $r$  are you sure it converges?

Try the following values:

$$r = 100$$

$$r = 1/4$$

$$r = 10$$

$$r = 0$$

$$r = 2$$

$$r = -1/2$$

$$r = 1$$

$$r = -1$$

$$r = 2/3$$

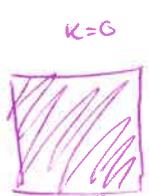
$$r = -10$$

$$r = 1/2$$

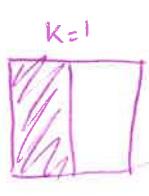
$$r = -100$$

Solution:

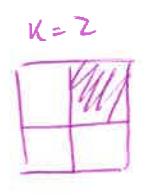
- talk about  
then prove
- For  $r > 1$ , we always add values at least 1, so it goes to infinity.
  - For  $r = 1/2$ : we add the following



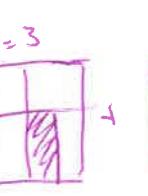
$$\text{K=0} \quad \text{K=1}$$



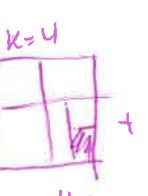
$$\text{K=1}$$



$$\text{K=2}$$



$$\text{K=3}$$



$$\text{K=4}$$

$$\left(\frac{1}{2}\right)^0 = 1$$

$$\left(\frac{1}{2}\right)^1 = \frac{1}{2}$$

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$\left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

The sum is the full square



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This means  $\lim_{n \rightarrow \infty} \sum_{k=0}^n \left(\frac{1}{2}\right)^k = 2$ .

- For  $r=0$ ,  $\underbrace{0^0 + 0^1 + 0^2 + 0^3 + \dots}_{1+0+0+\dots} = 1$

- For  $r=-1$ ,

$$\sum_{k=0}^n (-1)^k = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

The sum never converges.

- The same kind of argument holds for  $r < -1$ .

### Theorem

$\left\{ \sum_{k=0}^n r^k \right\}_{n \in \mathbb{N}}$  converges if  $-1 < r < 1$ , and diverges otherwise.

If  $-1 < r < 1$ ,

$$\lim_{n \rightarrow \infty} \left( \sum_{k=0}^n r^k \right) = \frac{1}{1-r}.$$

Sketch of the proof:

Verify that

$$(1-r) \lim_{n \rightarrow \infty} \left( \sum_{k=0}^n r^k \right) = 1, \quad \text{when } -1 < r < 1.$$

If it converges, this means that

$$\begin{aligned} 1 &= (1-r) + r(1-r) + r^2(1-r) + r^3(1-r) + r^4(1-r) + \dots \\ &= 1 - r + r - r^2 + r^2 - r^3 + r^3 - r^4 + r^4 + \dots \\ &= 1 - \lim_{n \rightarrow \infty} r^n \end{aligned}$$



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Opening

what could the following be?

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!}$$

Hint: It might be the limit of a MacLaurin polynomial.

References: James STEWART, Calculus, 8<sup>th</sup> edition.  
Sections 11.1 and 11.2.