

Recalls from preceding lectures:

- Taylor polynomials are local approximations of smooth functions
- Hopefully, the sequence  $\{T_n(x)\}_{n \in \mathbb{N}}$  converges to the function we want to approximate, at least around the value at which it is centered.

How good is the Taylor polynomial's approximation?

We define the n-th remainder of the Taylor polynomial as

$$R_n(x) = f(x) - T_n(x).$$

What do we hope from  $R_n(x)$ ? That it is small.

If  $\lim_{n \rightarrow \infty} R_n(x) = 0$ , then  $f(x) = \lim_{n \rightarrow \infty} T_n(x)$

But when  $n$  is small, we want to know what the error could be.

Intuitively, the approximation is very good in a given radius close to the point  $a$  where we approximate.

As we go further from  $a$ , the error gets much bigger, and can even rocket.

We describe the error margin within a given radius.

Theorem (Taylor's error formula) If  $|f^{(n+1)}(x)| \leq M$  for  $|x-a| \leq d$ , then the remainder  $R_n(x)$  within distance  $d$  from  $a$

of  $T_n(x)$  satisfies

$$|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}, \text{ for } |x-a| \leq d.$$

Please note that we do not expect you to memorize this formula, only to be able to use it.

## What it means?

- $T_n(x)$  approximates  $f(x)$  when  $x$  is close to  $a$ .
- $M$  is an upper bound on the  $(n+1)$ -st derivative in a radius  $d$  around  $a$ .
- In that radius, the error is bounded above by  $\frac{M|x-a|^{n+1}}{(n+1)!}$ .

### Example

$f(x) = \sin(x)$ , around  $a=0$ .

- The 1<sup>st</sup>, 5<sup>th</sup>, 9<sup>th</sup>, 13<sup>th</sup>, ... derivative of  $f$  is  $\cos(x)$  (1, when evaluated in  $x=0$ )
- The 3<sup>rd</sup>, 7<sup>th</sup>, 11<sup>th</sup>, 15<sup>th</sup>, ... derivative of  $f$  is  $-\cos(x)$  (-1 in  $x=0$ )
- All even derivatives of  $f$  are 0

Then,

$$T_{15}(x) = \frac{x}{3!} - \frac{x^3}{5!} + \frac{x^5}{7!} - \frac{x^7}{9!} + \frac{x^9}{11!} - \frac{x^{11}}{13!} + \frac{x^{13}}{15!} - \frac{x^{15}}{17!}$$

Since  $|f^{(16)}(x)| = |\sin(x)| \leq 1$ , then

$$|R_{15}(x)| \leq \frac{x^{16}}{16!}$$

Note that here, we took the maximal value for  $f^{(16)}(x)$  over  $\mathbb{R}$ , but that is usually not possible. This is why we omit here to talk about  $d$ .

How big is  $\frac{x^{16}}{16!}$ ?

If  $x$  is small - let's say, smaller than 5, it is very very small.

It grows very fast!

Demo: plot  $T_{15}(x)$ ,  $\sin(x)$  and  $\frac{x^{16}}{16!}$  on the same picture, with Geogebra.

(3)

Question: Let  $T_n(x)$  be the Taylor approximation of  $e^x$  around  $a=1$ .

How good is  $T_3(x)$  to estimate  $e^{1.2}$ ?

5 minutes

2-3 people  
together.

Solution:

First, compute  $T_3(x)$ :

$$f(x) = e^x$$

$$f(1) = e$$

$$f'(x) = e^x$$

$$f'(1) = e$$

$$f''(x) = e^x$$

$$f''(1) = e$$

$$f'''(x) = e^x$$

$$f'''(1) = e$$

$$\Rightarrow T_3(x) = e + e(x-1) + \frac{e(x-1)^2}{2} + \frac{e(x-1)^3}{6}$$

How is the error at  $x=1.2$ ?

The maximum of  $f^{(4)}(x)$  in the range  $[0.8, 1.2]$  is  $e^{1.2} = M$ .

Then,

$$\begin{aligned}|R_3(1.2)| &\leq \frac{e^{1.2} (0.2)^4}{24} \\&= e^{1.2} \frac{0.0016}{24} \\&\stackrel{\text{as}}{\leq} e^2 \leq 10 \underbrace{\frac{0.0016}{24}}_{\leq 0.0001} \\&\leq 0.001.\end{aligned}$$

- Is the error:
- bigger than  $1/2$
  - in the range  $0.1$  to  $0.5$
  - in the range  $0.01$  to  $0.1$
  - " " "  $0.001$  to  $0.01$
  - smaller than  $0.001$ ?

Not mandatory  
to compute the  
error, but  
essential to  
estimate  $e^{1.2}$ .

Question: What is the maximal  $d$  so that the error of the 2<sup>nd</sup> approximation of  $\sin(x)$  around  $x=3$  is smaller than 0.02?

We know that  $|(\sin(x))^{(3)}| = |\cos(x)| \leq 1$ . (Also, around  $x=3$ ,  $\cos(x)$  should be close to 1. Any idea why?).

So set  $M=1$ .

Then,

$$|R_n(3+d)| \leq \frac{1 \cdot d^4}{24},$$

and we want  $\frac{d^4}{24} \leq 0.02$  the remainder to be smaller than 0.02.

$$\frac{d^4}{24} \leq 0.02$$

$$\Rightarrow d^4 \leq 0.48$$

$$\Rightarrow d \leq \sqrt[4]{0.48}$$

And the maximal  $d$  would be  $\sqrt[4]{0.48}$ .

(For your information, this is roughly 0.832, but we are not asking you this).

Reference: James STEWART, Calculus, 8<sup>th</sup> edition.  
Section 11.10.