Math 8- Lecture 4

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09/23/2019

Convergence of series.

Recall

A series is a sum of all the terms of a sequence.

Example

From the sequence $\{\frac{1}{2^n}\}_{n\in\mathbb{N}}$, we get the series $\sum_{n\in\mathbb{N}} \frac{1}{2^n} = 2$.

A geometric series is the sum of terms in the sequence $\{r^n\}_{n\in\mathbb{N}}$, for some value of r.

We saw on Wednesday that the series $\sum_{n\geq 0} r^n$ converges if and only if $-1<r<1$.

In general, how do we know that a series converges?

- Clearly, if the terms we are summing are not decreasing and positive, their sum is infinite.

Theorem

If the series $\sum_{n\geq 0} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Remark

This is equivalent to saying that, if $\lim_{n\to\infty} a_n \neq 0$ or if the limit does not exist, then $\sum_{n\geq 0} a_n$ diverges.

However, this is NOT equivalent to saying that if the limit is 0, then the series converges. (this is false, and there is a counterexample below).

Example

The series $\sum_{n\geq 1} \frac{1}{n}$ diverges, even if $\lim_{n\to\infty} \frac{1}{n} = 0$.

Hint to solve it: This is not that far from $\int_1^\infty \frac{1}{x}dx = \underbrace{\ln(\infty)}_{+\infty} - \ln(1)$...

Theorem (Ratio test)

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, the series $\sum_{n=0}^{+\infty} a_n$ converges.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ (or if it is infinite), the series $\sum_{n=0}^{\infty} a_n$ diverges.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive.

Example (Geometric series)

Using the test ratio, we can prove that geometric series $\sum_{n=0}^{\infty} r^n$ converges if and only if $-1 < r < 1$.

Example

Does the series $\sum_{n=0}^{\infty} \frac{n^3}{3^n}$ converges?

2 min
by themselves

Using the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}\right| = \lim_{n\to\infty} \frac{1}{3} \left|\frac{(n+1)^3}{n^3}\right|$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3$$

$$= 1/3 < 1$$

That series converges.

Example

Does the series $\sum_{n=1}^{\infty} 1/n^2$ converges?

1 min

Using the ratio test

$$\lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = 1.$$

The test is inconclusive. (However, the answer is yes).

Convergence of power series

Definition

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

coefficients

variable x

More generally, a power series centered at a is

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n + \dots$$

Example

Geometric series, when all coefficients are 1, are evaluations of $\sum_{n=0}^{\infty} x^n$ in $x=r$.

We then say that the power series $\sum_{n=0}^{\infty} x^n$ converges when $|x|<1$, or in a radius 1 from 0.

Theorem

Convergence of power series.

For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are only 3 possibilities

- The series converges only at $x=a$.
- The series converges for all $x \in \mathbb{R}$.
- There is a positive number R such that the series converges if $|x-a|<R$ and diverges if $|x-a|>R$. Such a number R is the radius of convergence, and the interval of convergence, is the set of all values of x for which the series converges.

Examples

- Geometric series

  $\sum_{n=0}^{\infty} x^n$ has a radius of convergence of $\underline{1}$, and its interval of convergence is $(-1,1)$

- The series $\sum_{n=0}^{\infty} n!\, x^n$ is a power series that converges only for $x=0$, and has thus a radius of convergence of 0.

- $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ has a radius of convergence of 1 (that can be seen through to the ratio test) and an interval of convergence of $[2,4)$

- $\sum_{n=1}^{\infty} \frac{x}{n!}$ has a radius of convergence of size ∞, and its interval of convergence is $(-\infty,\infty)$

Reference: James STEWART. Calculus, 8th edition.
Sections 11.2, 11.6 and 11.8.