

Recall from last lecture:

- We introduced power series and their radius of convergence.
- The goal of this is to represent functions, and what we can do with them.

How do we find power series?

- When they are not hard to compute, Taylor Series are great.
- Sometimes, derivatives and integrals can be more useful.

Theorem (term-by-term differentiation and integration)

If the power series  $\sum_{n \geq 0} c_n (x-a)^n$  has radius of convergence  $R > 0$ , the function  $f$  defined by

$$f(x) = \sum_{n \geq 0} c_n (x-a)^n$$

is differentiable (and therefore continuous) on the interval  $(a-R, a+R)$ , and

$$(i) f'(x) = c_1 + 2c_2(x-a)^2 + 3c_3(x-a)^3 + \dots = \sum_{n \geq 1} n c_n (x-a)^{n-1}$$

$$(ii) \int f(x) dx = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \dots$$

$$= C + \sum_{n \geq 0} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in (i) and (ii) are both  $R$ .

Question: Can you find the  $7^{\text{th}}$  Taylor polynomial of  $\tan^{-1}(x)$  at  $a=0$ ?  
 or  $\arctan(x)$

Doing the regular way for Taylor polynomials, we would get the following:

$$f(x) = \tan(x)$$

$$f'(x) = \frac{1}{1+x^2} *$$

$$f''(x) = \frac{-2x}{(1+x^2)^2}$$

$$f'''(x) = \frac{-2(1+x^2) - 4x}{(1+x^2)^3}$$

Explanation of  $f'(x) = \frac{1}{1+x^2}$ :

$$\arctan(x) = y$$

$$\Rightarrow x = \tan(y)$$

$$\Rightarrow (x)' = (\tan(y))'$$

$$\Rightarrow (= \sec^2(y) \cdot y')$$

$$\Rightarrow (= (1+x^2) \cdot y')$$

$$\Rightarrow y' = \frac{1}{1+x^2}.$$



(deduced from  
 $x = \tan(y)$ .)

: Of course, many things cancel each other out, but it still seems painful! And we are not still close to the  $7^{\text{th}}$  Taylor polynomial!

Using the theorem,

$$\tan^{-1}(x) = \int \frac{1}{1+x^2} dx$$

$$= \int \sum_{n=0}^{\infty} x^{2n} dx$$

because this is a geometric series.

$$= \int (1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \dots) dx \quad \text{expansion}$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \dots$$

term-by-term integration.

To find  $C$ , we evaluate in  $x=0$ :  $\tan^{-1}(0)=0 \Rightarrow C=0$ , in this case Therefore,

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1},$$

in the interval of convergence of  $\frac{1}{1+x^2}$ , which is  $(-1, 1)$  (it is geometric).

## Quick review of integration

- Do you remember the Fundamental Theorem of Calculus?

A version of it is that, if  $f'$  is continuous on  $[a, b]$ , then

$$\int_a^b f'(x) dx = f(b) - f(a).$$

What does  $f'(x)$  mean? It is the rate of change of  $f$ . This lead to the following theorem:

Theorem (Net change theorem).

The integral of a rate of change is the net change:

$$\int_a^b f'(x) dx = f(b) - f(a).$$

### Example

Let  $V'(t)$  be the rate at which water flows into a reservoir at time  $t$ . Then, the net change in the volume at times  $t_1$  and  $t_2$  is

$$\int_{t_1}^{t_2} V'(t) dt = V(t_2) - V(t_1).$$

- Substitution rule.

In teams of 2-3 people, solve the following:

$$\bullet \int x^3 \cos(x^4+2) dx \quad \bullet \int \sqrt{2x+1} dx.$$

For both these integrals, we need the substitution theorem.

Theorem (Substitution rule).

If  $u=g(x)$  is differentiable and  $f$  is continuous, then

$$\int f(g(x)) g'(x) dx = \int f(u) du.$$

Solutions to the two integrals:

(4)

- $\int x^3 \cos(x^4+2) dx$

can be solved by taking  $u = x^4 + 2$ ,  $du = 4x^3 dx$

$$\begin{aligned}\int x^3 \cos(x^4+2) dx &= \int \frac{1}{4} \cos(u) du \\ &= \frac{1}{4} \sin(u) + C. \\ &= \frac{1}{4} \sin(x^4+2) + C\end{aligned}$$

- $\int \sqrt{2x+1} dx$  will be solved with  $u = 2x+1$ ,  $du = 2dx$ .

$$\begin{aligned}\int \sqrt{2x+1} dx &= \int \frac{1}{2} \sqrt{u} du \\ &= \frac{u^{3/2}}{3} + C \\ &= \frac{(2x+1)^{3/2}}{3} + C,\end{aligned}$$

Reference: James STEWART. Calculus, 8th edition.  
Sections 4.4, 4.5 and 11.9.