

Integrals as volumes and areas

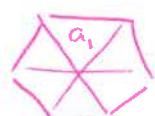
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Last class

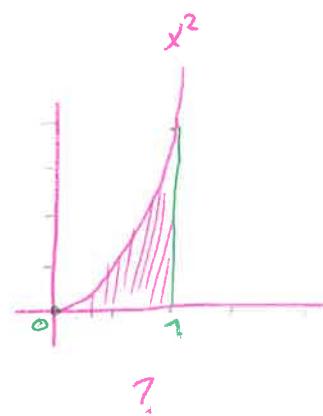
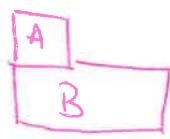
- we differentiated and integrated some sums, by considering the integral as the antiderivative.

For this class, instead of using integrals to solve some sum's problems, we will define the integral as the limit of a given sum:

Question: How do we compute the area of a definite region?

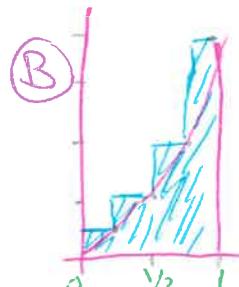
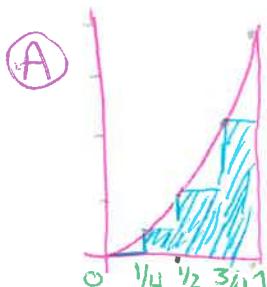


a regular hexagon can be cut into regular triangles



Can we decompose the area underneath a curve into rectangles / triangles?

Example: dividing the area below x^2 between 0 and 1 into 4 rectangles:



What are the problems?

- (A) is too small; but (B) is too high.

Computing the estimated area:

$$(A) \text{ width of the 1st rectangle} \frac{1}{4} \cdot \text{height of 1st rectangle} (0)^2 + \frac{1}{4} \left(\frac{1}{4}\right)^2 + \frac{1}{4} \left(\frac{2}{4}\right)^2 + \frac{1}{4} \left(\frac{3}{4}\right)^2 = \frac{\pi}{32}$$

$$(B) \frac{1}{4} \left(\frac{1}{4}\right)^2 + \frac{1}{4} \left(\frac{2}{4}\right)^2 + \frac{1}{4} \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot 1^2 = \frac{15}{32}.$$

How can we improve?

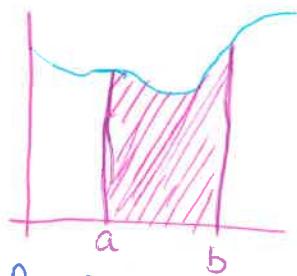
- * We took the lowest (respectively highest) value of x^2 in a given interval to compute the area in (A) (resp. (B)). Could we take something in the middle? This is however not so well defined if the function is not monotone.
- * If we take more rectangles, the estimate should be more precise.

Riemann sums

To compute the area under a function f , over an interval $[a, b]$:

- 1- Divide $[a, b]$ into many segments of width $\frac{b-a}{n}$.
- 2- In the k -th interval, choose a point x_k .
- 3- For each interval, draw a rectangle of width $\frac{b-a}{n}$ such that the k -th is of height $f(x_k)$
- 4- Approximate the total area by the sum of the areas of the rectangle. The latter is called the Riemann sum.

If all x_k are taken on the left (respectively right) end of the interval, the sum is called the left Riemann sum (resp. right Riemann sum)



(3)

Theorem

If the function f is integrable on $[a, b]$, the limit of the left and right Riemann sums are the same, and

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \underbrace{\left(\frac{b-a}{n}\right)}_{\substack{\text{width} \\ \text{of a} \\ \text{rectangle}}} \underbrace{f(x_k)}_{\substack{\text{height of} \\ \text{the } k\text{th} \\ \text{rectangle}}} = \int_a^b f(x) dx.$$

This is the definite integral.

Example

Estimating the area under $f(x) = x^2$ between 0 and 1 by the right Riemann sum gives

$$\begin{aligned} \sum_{k=1}^n \frac{1}{n} \left(\frac{k}{n}\right)^2 &= \sum_{k=1}^n \frac{k^2}{n^3} \\ &= \frac{1}{n^3} \sum_{k=1}^n k^2 \\ &= \frac{2n^3 + 3n^2 + n}{6n^3}. \end{aligned}$$

Claim

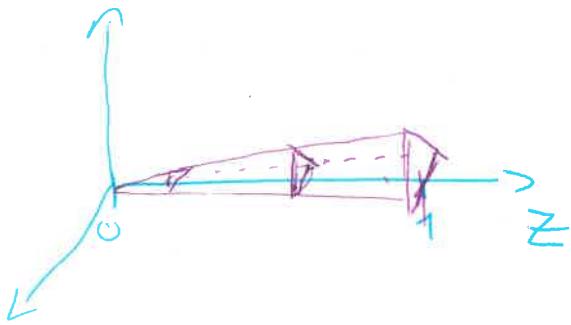
$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

As $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \left(\frac{k}{n}\right)^2 = \lim_{n \rightarrow \infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{2}{6} = \frac{1}{3}.$$

This is coherent with our definition of the integral as the anti derivative: $\int_0^1 x^2 dx = \left(\frac{x^3}{3}\right) \Big|_{x=0}^{x=1} = \frac{1}{3}$.

Question: Can you use the technique of the Riemann sum to compute the volume of the tetrahedron that is such that the projection in any $z \in [0, 1]$ is an equilateral triangle of base z .



We cut the $[0, 1]$ -interval on z into n slices that are like thick triangular prisms, of thickness $\frac{1}{n}$.

Taking the right Riemann sum, the k -th prism is made out of a triangle of base $k\frac{1}{n}$, and thus of area $\frac{\sqrt{3} k^2}{4n^2}$.

The Riemann sum is

$$\sum_{k=1}^n \frac{1}{n} \cdot \frac{\sqrt{3} k^2}{4n^2} = \frac{\sqrt{3}}{4n^3} \sum_{k=1}^n k^2 = \frac{\sqrt{3}}{4n^3} \cdot \left(\frac{2n^3 + 3n^2 + n}{6} \right).$$

$$\begin{aligned} k\frac{1}{n} / & \left(\frac{(k\frac{1}{n})^2}{4} \right) &= \frac{\sqrt{3} k}{2n} \\ & \underbrace{k\frac{1}{n}} \end{aligned}$$

by the claim on
the previous page.

As $n \rightarrow \infty$, we get closer to the volume:

$$\lim_{n \rightarrow \infty} \frac{\sqrt{3}}{4n^3} \left(\frac{2n^3 + 3n^2 + n}{6} \right) = \frac{\sqrt{3}}{12}.$$

(Animation with Geogebra).

Definition

Let S be a solid that lies between $x=a$ and $x=b$. If the cross-sectional area of S in the plane P_x , through x and perpendicular to the x -axis, follows a continuous function $A(x)$, then the volume of S is

$$V = \lim_{n \rightarrow \infty} \sum_{k=1}^n \underbrace{A(x_k)}_{\substack{x_k \text{ is in} \\ \text{the } k\text{-th interval}}} \frac{(b-a)}{n} = \int_a^b A(x) dx$$

Average value

Question: What is the average value of x^2 between 0 and 1?

- * We cannot take $(\frac{1-0}{2})^2$, because that does only take into account one point.
- * We can however take the area, and divide by the width.

Definition

The average value of an integrable (and therefore continuous) function for $[a,b]$ is

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Example

The average value of x^2 on $[0,1]$ is $\frac{1}{1-0} \left(\frac{1^3}{3} - \frac{0^3}{3} \right) = \frac{1}{3}$.

(6)

Reference : James STEWART. Calculus, 8th edition.

§ 4.1, 4.2, 5.2 and 5.5