

Work

The lecture today is about some application of Riemann sums and integrals to physics. More specifically, we will explore the notion of work, which corresponds more or less to the quantity of effort required to perform a task.

Definition

In physics, the work is the product of force and displacement.

A force is some quantity that, when applied on an immobile object, makes the object move.

Example of forces

- Gravity
- Friction
- Spring compression.
- Pulling on an object.

Work when the force is constant, and so is the displacement

When the force is constant, we can simply measure the work by multiplying this force by the total displacement. (Just like we compute the area of a rectangle by multiplying the base and the height.)

In the SI (metric) system, the work is measured in joules (J), the displacement in meters (m), the time in seconds (s), the force in newtons ($N = \text{kg} \cdot \text{m/s}^2$).

$$1 \text{ J} = 1 \text{ N} \cdot \text{m} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^2$$

$$W = F \cdot s$$

Computing the force

Newton's Second law of motion

The force applied on an object to make it move is the product of its mass and its acceleration:

$$F = m \cdot a$$

(2)

Close to the surface of the Earth, the gravity is computed using an acceleration of $g=9.8 \text{ m/s}^2$. (You are not expected to remember it. We will tell you if that is relevant).

Example 1

How much work is done in lifting a 1.2-kg book off the floor to a desk that is 0.7 m high?

1- Compute the force

$$F = m \cdot a,$$

Where the mass is 1.2 kg and the acceleration is $g=9.8 \text{ m/s}^2$

Thus, the force is $(1.2 \times 9.8) \text{ N} = 11.76 \text{ N}$.

2- Compute the work

$$W = F \cdot s$$

↳ displacement

The displacement here is 0.7 m. Hence, $W = 11.76 \text{ N} \cdot 0.7 \text{ m}$

$$\begin{aligned} &= (11.76 \cdot 0.7) \text{ J} \\ &= 8.232 \text{ J}. \end{aligned}$$

Work when the force or the displacement is not constant.

We split the displacement into a Riemann sum, and we solve the problem doing an integral.

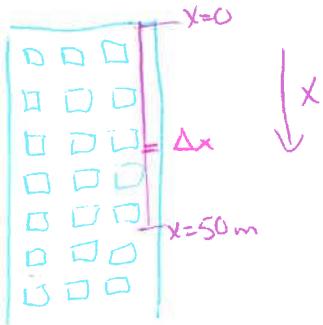
Example

A 100-kg cable is 50m long and hangs vertically from a tall building. How much work is required to lift the cable to the top of the building?

Here, the work is not constant along the cable, because the displacement is not the same.

We cut the cable into very thin pieces.

③



We divide the cable into parts of length $\Delta x = \frac{50}{n}$. From the top, the total displacement of the i -th part is $\frac{50 \cdot i}{n}$. The force is always given by the gravity, and this is $m \cdot 9.8 \text{ m/s}^2$, where the mass is $\frac{200}{n}$.

Hence, the work done by the i -th part is

$$W_i = 9.8 \cdot \frac{200}{n} \cdot \frac{50i}{n}$$

$\underbrace{}_F \quad \underbrace{}_S$

and the total work is $W = \sum_{i=1}^n W_i$.

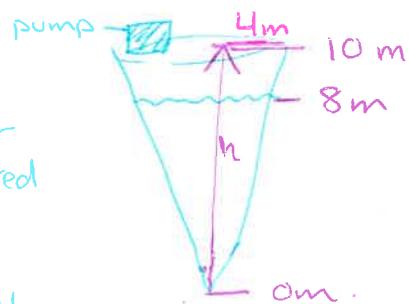
While $n \rightarrow \infty$, we can transform it into an integral,

with $dx = \frac{50}{n}$ and $x = \frac{50i}{n}$

$$W = \int_0^{50} 9.8 \cdot 4 \cdot x \, dx = \left(\frac{39.2}{2} x^2 \right) \Big|_0^{50} = \frac{39.2 \cdot 2500}{2} = 49000 \text{ J}$$

Example

A tank has the shape of an inverted circular cone with height 10m and base radius 4m. It is filled with water to the height of 8m. Find the work required to empty the tank by pumping all of the water to the top of the tank. The density of the water is 1000 kg/m^3 .



Here, the displacement of the water depends on where it was originally in the tank.

We cut the cone into disks of height $\Delta h = \frac{10}{n}$ and of radius

$$\frac{4}{10} h = \frac{4}{10} \cdot \frac{10i}{n} = \frac{4i}{n}. \text{ The volume of the } i\text{-th disk is } \left(\frac{4i}{n}\right)^2 \pi \left(\frac{10}{n}\right) = \frac{160\pi i^2}{n^3} \text{ m}^3.$$

Force:

The force is the multiplication of the mass and the gravitational acceleration (9.8 m/s^2), and the weight of the i -th disk

$$\frac{1000 \cdot 160\pi i^2}{n^3} \cdot 9.8 \text{ kg.}$$

1000 : density
 $\frac{160\pi i^2}{n^3}$: volume

9.8 : acceleration

Displacement:

The displacement of the water in the i -th disk is

$$10-h = 10 - \frac{10i}{n}$$

Work

The total work done by the water in the i -th disk is

$$W_i = \frac{1000 \cdot (160\pi i)^2 \cdot 9.8}{n^3} \left(10 - \frac{10i}{n}\right), \text{ for } \frac{10i}{n} \leq 8.$$

and the total work is

$$\sum_{i=1}^n W_i.$$

because there is no water above that.

As $n \rightarrow \infty$, we replace $h = \frac{10i}{n}$, $\Delta h = \frac{10}{n}$, and

$$\begin{aligned} W &= \int_0^8 h^2 \cdot 160\pi 9.8 \left(10 - \frac{h}{10}\right) dh \\ &= 160 \cdot 9.8 \pi \int_0^8 10h^2 - h^3 dh \\ &= 160 \cdot 9.8 \cdot \pi \left(10 \cdot \frac{8^3}{3} - \frac{8^4}{4}\right). \end{aligned}$$

$$\approx 3.36 \times 10^6 \text{ J.}$$

Question: Two large tanks are shaped like square pyramid with height 10 m and base 5m by 5m. One is upright, the other one is inverted. Which tank takes more work to empty if the pump is at the top? Hint: Think of the total displacement.

Reference: James STEWART. Calculus, 8th edition. Section 5.4.