Math 8
Fall 2019
Section 2
October 11, 2019

First, some important points from the last class:

Vectors are used to model anything that has magnitude (represented by the length of the vector) and direction (represented by the direction of the vector).

Example: The displacement of an object moving from point $P = (a_1, b_1, c_1)$ to point $Q = (a_2, b_2, c_2)$ is represented by the displacement vector $\langle a_2 - a_1, b_2 - b_1, c_2 - c_1 \rangle$.

Vectors, vector addition, scalar multiplication, subtraction, magnitude (norm), and direction have algebraic and geometric representations, and different meanings in different applications.

Two arrows with the same length and direction are two pictures of the same vector.

Vector addition and subtraction follow parallelogram laws.

The norm (magnitude) of $\langle a, b, c \rangle$ is

$$| \langle a, b, c \rangle | = \sqrt{a^2 + b^2 + c^2}$$

Direction is generally given by a unit vector, a vector whose norm is 1.

Parallel vectors have the same or opposite directions; each is a scalar multiple of the other.

Theorem: If $\vec{v}$ is a nonzero vector, the unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \frac{1}{|\vec{v}|}\vec{v}.$$

Standard basis for $\mathbb{R}^3$:

$$\{\hat{i}, \hat{j}, , \hat{k}\}$$

$$\hat{i} = \langle 1, 0, 0 \rangle$$

$$\hat{j} = \langle 0, 1, 0 \rangle$$

$$\hat{k} = \langle 0, 0, 1 \rangle$$

$$\langle a, b, c \rangle = a\hat{i} + b\hat{j} + c\hat{k}$$

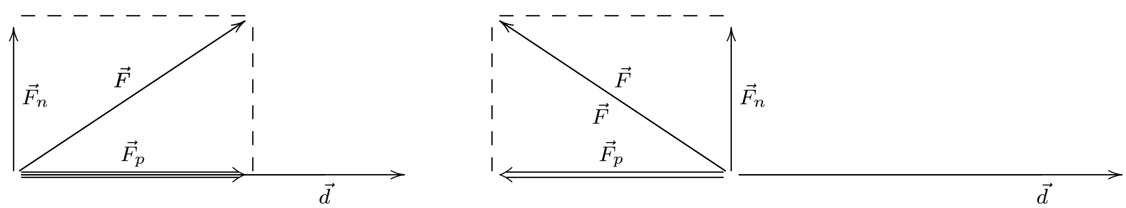
Notation: Sometimes vectors are written with an arrow on top, $\vec{v}$. Sometimes, instead, they are written in boldface, $\mathbf{v}$.

Sometimes the norm of the vector $\vec{v}$ is written $||\vec{v}||$ instead of $|\vec{v}|$.

It is common, particularly in physics and engineering, to write the vector $\langle a, b, c \rangle$ as $a\hat{i} + b\hat{j} + c\hat{k}$, or as $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, or as $a\hat{x} + b\hat{y} + c\hat{z}$.

You may recall "work equals force times distance." However, it isn't quite that simple. If the force acts in the opposite direction to the motion, it does negative work. (That is, work is done against the force.) If the force is perpendicular to the direction of motion, it does no work at all. If the force is at an angle to the direction of motion, the work depends on the component of the force parallel to the direction of motion.

An object acted on by force $\vec{F}$ moves in a straight line along vector $\vec{d}$. (There are also other forces acting on the object.) We are interested in the work done by $\vec{F}$. Here are two possible pictures.



The force $\vec{F}$ can be expressed as the sum of two parts, $\vec{F}_p$ parallel to the direction of motion, and $\vec{F}_n$ normal (or perpendicular, or orthogonal) to the direction of motion. Only $\vec{F}_p$ does work on the moving object.

The vector $\vec{F}_p$ is called the *vector projection* of $\vec{F}$ onto $\vec{d}$ (or in the direction of $\vec{d}$).

It is sometimes denoted $proj_{\vec{d}}(\vec{F})$ and called simply the projection of $\vec{F}$ onto $\vec{d}$.

The work done by $\vec{F}$ on our moving object is

$$W = \begin{cases} |\vec{F}_p|\,|\vec{d}| & \text{if } \vec{F}_p \text{ has the same direction as } \vec{d}; \\ \\ -|\vec{F}_p|\,|\vec{d}| & \text{if } \vec{F}_p \text{ has the opposite direction from } \vec{d}. \end{cases}$$

The *component* of $\vec{F}$ along $\vec{d}$ (or the scalar projection of $\vec{F}$ onto $\vec{d}$), sometimes denoted $comp_{\vec{d}}(\vec{F})$, is the quantity

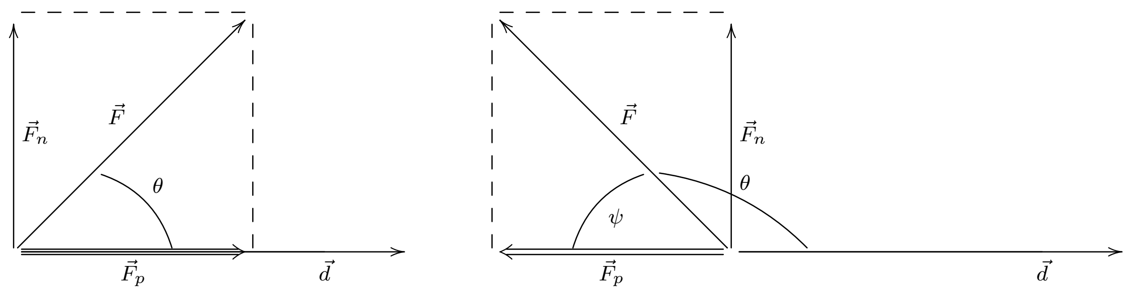
$$\begin{cases} |\vec{F}_p| & \text{if } \vec{F}_p \text{ has the same direction as } \vec{d}; \\ \\ -|\vec{F}_p| & \text{if } \vec{F}_p \text{ has the opposite direction from } \vec{d}. \end{cases}$$

$$W = \left(comp_{\vec{d}}(\vec{F})\right)|\vec{d}|$$

It is sometimes called the coordinate of $\vec{F}$ in the direction of $\vec{d}$.

"Work equals (component of force in the direction of motion) times distance."

Finding vector and scalar projections:



If $\vec{F}$ makes an angle of θ with $\vec{d}$, then the component of $\vec{F}$ along $\vec{d}$ is

$$comp_{\vec{d}}(\vec{F}) = |\vec{F}|\cos(\theta)$$

and in our case, the work done by $\vec{F}$ on our object is the component of $\vec{F}$ in the direction of motion times the distance moved,

$$W = comp_{\vec{d}}(\vec{F})|\vec{d}| = |\vec{F}|\,|\vec{d}|\,\cos(\theta).$$

The projection of $\vec{F}$ onto $\vec{d}$ has the same direction as $\vec{d}$ if the component is positive, and the opposite direction if the component is negative.

Let $\vec{u}$ be a unit vector in the direction of $\vec{d}$,

$$\vec{u} = \frac{1}{|\vec{d}|}\,\vec{d}.$$

Then the projection of $\vec{F}$ onto $\vec{d}$ has the same direction as $\vec{u}$ if the component is positive, and the opposite direction if the component is negative. Also its length is the absolute value of the scalar projection.

Putting all this together, the projection of $\vec{F}$ onto $\vec{d}$ is

$$proj_{\vec{d}}(\vec{F}) = \Big(comp_{\vec{d}}(\vec{F})\Big)\vec{u} = \Big(|\vec{F}|\cos(\theta)\Big)\vec{u} = \left(\frac{|\vec{F}|\cos(\theta)}{|\vec{d}|}\right)\vec{d}$$

For future reference:

$$W = \boxed{|\vec{F}|\,|\vec{d}|\,\cos(\theta)} \qquad comp_{\vec{d}}(\vec{F}) = |\vec{F}|\cos(\theta) = \frac{\boxed{|\vec{F}|\,|\vec{d}|\cos(\theta)}}{|\vec{d}|}$$

$$proj_{\vec{d}}(\vec{F}) = \left(\frac{|\vec{F}|\cos(\theta)}{|\vec{d}|}\right)\vec{d} = \left(\frac{\boxed{|\vec{F}|\,|\vec{d}|\cos(\theta)}}{|\vec{d}|^2}\right)\vec{d}$$

There is a kind of product of vectors that helps us to compute these projections. The *dot product* of two vectors is a scalar, defined by

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}|\,\cos(\theta),$$

where θ is the angle between $\vec{v}$ and $\vec{w}$. (We'll see an algebraic formula for the dot product shortly.)

The dot product is sometimes called the *scalar product*, because the dot product of two vectors is a scalar. This is not the same as scalar multiplication.

Example: If $\vec{v}$ and $\vec{w}$ are perpendicular to each other, then

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos\left(\frac{\pi}{2}\right) = 0.$$

If $\vec{v}$ and $\vec{w}$ have the same direction,

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos(0) = |\vec{v}|\,|\vec{w}|,$$

and if they have opposite directions,

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos(\pi) = -|\vec{v}|\,|\vec{w}|.$$

In particular,

$$\vec{v} \cdot \vec{v} = |\vec{v}|^2.$$

We can use the dot product to rewrite our formulas:

If $\vec{F}$ makes an angle of θ with $\vec{d}$, then the component of $\vec{F}$ along $\vec{d}$ is

$$comp_{\vec{d}}(\vec{F}) = |\vec{F}| \cos(\theta) = \frac{|\vec{F}|\,|\vec{d}| \cos(\theta)}{|\vec{d}|} = \boxed{\frac{\vec{F} \cdot \vec{d}}{|\vec{d}|}}$$

and the projection of $\vec{F}$ onto $\vec{d}$ is

$$proj_{\vec{d}}(\vec{F}) = \left(\frac{|\vec{F}| \cos(\theta)}{|\vec{d}|}\right) \vec{d} = \left(\frac{|\vec{F}|\,|\vec{d}| \cos(\theta)}{|\vec{d}|^2}\right) \vec{d} = \boxed{\left(\frac{\vec{F} \cdot \vec{d}}{\vec{d} \cdot \vec{d}}\right) \vec{d}}$$

The work done by force $\vec{F}$ on an object moving in a straight line with displacement $\vec{d}$ is

$$\boxed{W = \vec{F} \cdot \vec{d}}$$

"Work equals force dot displacement."

Algebra of the dot product:

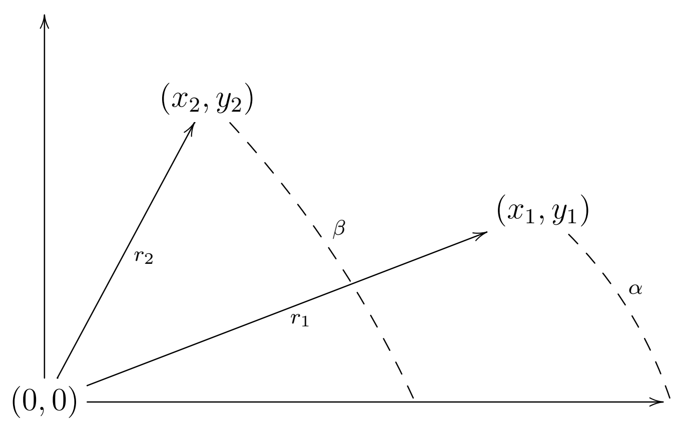
$$\boxed{\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3}$$

In particular,

$$\langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle = a_1^2 + a_2^2 + a_3^2 = |\,\langle a_1, a_2, a_3 \rangle\,|^2.$$

The corresponding formulas work in $\mathbb{R}^2$ (and in $\mathbb{R}^n$) as well.

Example: We can use this to solve the third part of the preliminary homework problem much more easily:



Let θ be the angle between the vector $\vec{v}_1$ from $(0,0)$ to (x_1, y_1) and the vector v_2 from $(0,0)$ to (x_2, y_2). Express the cosine of θ in terms of x_1, x_2, y_1, y_2, r_1 (the length of $\vec{v}_1$), and r_2 (the length of $\vec{v}_2$).

$$\vec{v}_1 = \langle x_1, y_1 \rangle \qquad \vec{v}_2 = \langle x_2, y_2 \rangle$$

$$\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1|\,|\vec{v}_2| \cos\theta$$

$$\cos\theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1|\,|\vec{v}_2|} = \frac{x_1 x_2 + y_1 y_2}{r_1 r_2}$$

Theorem (basic facts about dot products):

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}|\cos(\theta),$$

where θ is the angle between $\vec{v}$ and $\vec{w}$.

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

$$\vec{v} \cdot (\vec{w} + \vec{u}) = (\vec{v} \cdot \vec{w}) + (\vec{v} \cdot \vec{u})$$

$$\vec{v} \cdot (\vec{w} - \vec{u}) = (\vec{v} \cdot \vec{w}) - (\vec{v} \cdot \vec{u})$$

$$(t\vec{v}) \cdot \vec{w} = t(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (t\vec{w})$$

$$\vec{0} \cdot \vec{v} = 0$$

$$\vec{v} \cdot \vec{v} = |\vec{v}|^2$$

Theorem (the triangle inequality):

$$|\vec{v} + \vec{w}| \leq |\vec{v}| + |\vec{w}|$$

Theorem (the Cauchy-Schwartz inequality):

$$|\vec{v} \cdot \vec{w}| \leq |\vec{v}|\,|\vec{w}|$$

Example: A 2 kilogram box is dragged for 20 meters along a horizontal floor, by a rope making a 45 degree angle with the floor. If the force on the rope is 10 newtons, how much work does that force do on the box?

Questions:

Why do we know the triangle inequality

$$|\vec{v} + \vec{w}| \leq |\vec{v}| + |\vec{w}|$$

is true? (You may reason algebraically or geometrically.)

Why do we know the Cauchy-Schwartz inequality

$$|\vec{v} \cdot \vec{w}| \leq |\vec{v}|\,|\vec{w}|$$

is true?

We know that, if $\vec{v}$ and $\vec{w}$ are nonzero vectors, then $\vec{v} \cdot \vec{w} = 0$ means that $\vec{v}$ and $\vec{w}$ are orthogonal (perpendicular to each other). What does $\vec{v} \cdot \vec{w} > 0$ mean geometrically?

The first theorem on the previous page includes a commutative law for dot products:

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

Is there an associative law for dot products, $(\vec{v} \cdot (\vec{u} \cdot \vec{w}) = (\vec{v} \cdot \vec{u}) \cdot \vec{w})$? Why or why not?

Exercise: Find the projection of $\vec{v}$ on $\vec{w}$ and the component of $\vec{v}$ along $\vec{w}$ if

$$\vec{v} = \langle -1, -1, -2 \rangle \qquad \vec{w} = \langle 3, 4, 12 \rangle .$$

Exercise: TRUE or FALSE?

> If $\vec{u}$ is a unit vector, then the component of $\vec{v}$ along $\vec{u}$ is just $\vec{v} \cdot \vec{u}$, and the projection of $\vec{v}$ on $\vec{u}$ is just $(\vec{v} \cdot \vec{u})\,\vec{u}$.

Explain.

Exercise:

1. What does it mean geometrically if

$$(\vec{v} + \vec{w}) \cdot (\vec{v} - \vec{w}) = 0?$$

2. Use the basic facts about dot products to show that

$$(\vec{v} + \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 - |\vec{w}|^2$$

3. Use (1) and (2) to derive a theorem about the diagonals of a parallelogram.

Some solutions from last time:

Exercise: We set up our axes with the z-axis vertical, the x-axis pointing east, and the y-axis pointing north. An airplane's velocity relative to the air around it is $\langle 150, 200, 5\rangle$, with units of miles and hours. If the wind is blowing from the northeast at 15 miles per hour (and parallel to the ground), what is the airplane's velocity relative to the ground?

The velocity of the air relative to the ground has the direction of the vector $\langle -1, -1, 0\rangle$ (which points from the northeast and parallel to the ground) and magnitude 15 mph, so it is $\left\langle -\frac{15\sqrt{2}}{2}, -\frac{15\sqrt{2}}{2}, 0\right\rangle$. To find the velocity of the plane relative to the ground, we add the velocity of the plane relative to the air and the velocity of the air relative to the ground: $\left\langle 150 - \frac{15\sqrt{2}}{2}, 200 - \frac{15\sqrt{2}}{2}, 5\right\rangle$ mph.

Exercise: This example uses SI units of meters and seconds. Find a vector that represents the velocity of an object moving in a straight line from the origin toward the point $(15, 20, 60)$ (where coordinates are in meters) at a speed of .5 meters per second. (Suggestion: First find any vector in the direction of motion. Then find the velocity vector.)

One vector in the direction of motion is the displacement vector from the origin to $(15, 20, 60)$, or $\langle 15, 20, 60\rangle$. This vector has length $\sqrt{15^2 + 20^2 + 60^2} = 65$. To find a unit vector in this direction we can take $\left\langle \frac{15}{65}, \frac{20}{65}, \frac{60}{65}\right\rangle = \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right\rangle$.

The velocity vector should be pointing in this same direction but have magnitude .5, so we multiply this unit vector by .5 to get $\left\langle \frac{3}{26}, \frac{4}{26}, \frac{12}{26}\right\rangle \frac{\text{m}}{\text{s}}$.

If the object starts from the origin at time $t = 0$, where will it be at time $t = 6$? (Suggestion: Find its displacement, by finding its direction of motion and the distance it travels.)

The direction of the object's displacement is the direction of the velocity vector, which is the unit vector $\left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right\rangle$ we found above.

The distance the object moves is the speed .5 meters per second times the elapsed time 6 seconds, or 3 meters. Its displacement is 3 times the unit vector in the direction of motion, or $3\left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right\rangle = \left\langle \frac{9}{13}, \frac{12}{13}, \frac{36}{13}\right\rangle$ (meters).

The object's new position is its original position plus its displacement. In this case, the original position is at the origin, $\langle 0, 0, 0\rangle$, so its new position is $\left\langle \frac{9}{13}, \frac{12}{13}, \frac{36}{13}\right\rangle$ (meters).

Note: The object's displacement is the product of the time it travels, 6 seconds, and its velocity, $\left\langle \frac{3}{26}, \frac{4}{26}, \frac{12}{26}\right\rangle \frac{\text{m}}{\text{s}}$. This is not an accident.