Math 8
Fall 2019
Section 2
October 23, 2019

First, some important points from the last class:

Definition: If a curve γ has a regular parametrization $\vec{r} : [a, b] \to \mathbb{R}^n$ that does not retrace any portion of γ, then the arc length of γ is

$$L = \int_a^b |\vec{r}\,'(t)| \, dt.$$

The arc length function is the function that takes t to the arc length of the portion of γ between $\vec{r}(a)$ and $\vec{r}(t)$:

$$s(t) = \int_a^t |\vec{r}\,'(u)| \, du.$$

The parametrization of γ by arc length is the function that takes a number s to the point on γ that is a distance of s units along γ from the starting point $\vec{r}(a)$.

Compute this by using the arc length function (expressing s as a function of t) to instead express t as a function of s, say $t = f(s)$, then rewriting the expression $\vec{r}(t)$ by rewriting t in terms of s, that is, by setting $\vec{p}(s) = \vec{r}(f(s))$.

Definition: The curvature of a curve γ with regular parametrization $\vec{r}$ at a point $\vec{r}(t)$ is the magnitude of the rate of change of the unit tangent vector $\vec{T}$ with respect to arc length,

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| = \frac{1}{\left| \frac{ds}{dt} \right|} \left| \frac{d\vec{T}}{dt} \right|.$$

A curve of curvature κ bends as much as a circle of radius $\dfrac{1}{\kappa}$.

Theorem: The arc length and curvature of a curve can be computed using any regular parametrization, and the answer will be the same.

We say arc length and curvature do not depend on the parametrization.

By an earlier theorem, because $|\vec{T}|$ is constant, we have $\dfrac{d\vec{T}}{dt} \perp \vec{T}$. We define the *unit normal vector* $\vec{N}$ to be the unit vector in this direction, which points in the direction in which the curve bends:

$$\vec{N} = \frac{1}{\left| \frac{d\vec{T}}{dt} \right|} \frac{d\vec{T}}{dt} \qquad \frac{d\vec{T}}{dt} = \left| \frac{d\vec{T}}{dt} \right| \vec{N}$$

Analyzing acceleration:

Given the position $\vec{r}$ of an moving object as a function of the time t, we can compute:

$$\vec{v} = \frac{d\vec{r}}{dt} = \text{ velocity} \qquad \frac{ds}{dt} = |\vec{v}| = \text{ speed} \qquad \vec{v} = |\vec{v}|\,\vec{T} = \frac{ds}{dt}\,\vec{T}$$

$$\vec{T} = \frac{1}{\frac{ds}{dt}}\vec{v} = \text{ unit tangent vector } = \text{ unit vector in direction of motion}$$

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{\frac{ds}{dt}}\left|\frac{d\vec{T}}{dt}\right| = \text{ curvature} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \text{ acceleration}$$

$$\vec{N} = \frac{1}{\left|\frac{d\vec{T}}{dt}\right|}\frac{d\vec{T}}{dt} = \text{ unit normal vector} \qquad \frac{d\vec{T}}{dt} = \left|\frac{d\vec{T}}{dt}\right|\vec{N}$$

Using the product rule, we can write

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\vec{T}\right) = \left(\frac{d}{dt}\frac{ds}{dt}\right)\vec{T} + \frac{ds}{dt}\frac{d\vec{T}}{dt} = \underbrace{\left(\frac{d^2s}{dt^2}\right)}_{a_{\mathbf{T}}}\vec{T} + \underbrace{\frac{ds}{dt}\left|\frac{d\vec{T}}{dt}\right|}_{a_{\mathbf{N}}}\vec{N}$$

The acceleration $\vec{a}$ is expressed as the sum of two parts, one in the direction of motion, and one normal (perpendicular) to the direction of motion.

The scalar $a_{\mathbf{T}}$ is the tangential component of the acceleration. It equas $\dfrac{d^2s}{dt^2}$, the second derivative of distance with respect to time. We may call $\dfrac{d^2s}{dt^2}$ the linear acceleration. The tangential part of acceleration $a_{\mathbf{T}}\vec{T}$, is the part of acceleration in the direction of motion, or the projection of acceleration in the direction of $\vec{T}$.

The scalar $a_{\mathbf{N}}$ is the normal component of the acceleration. It equals

$$\frac{ds}{dt}\left|\frac{d\vec{T}}{dt}\right| = \frac{ds}{dt}\left|\frac{ds}{dt}\frac{d\vec{T}}{ds}\right| = \left(\frac{ds}{dt}\right)^2\left|\frac{d\vec{T}}{ds}\right| = \left(\frac{ds}{dt}\right)^2\kappa,$$

the speed squared times the curvature. The normal part of acceleration $a_{\mathbf{N}}\vec{N}$, is the part of acceleration perpendicular to the direction of motion, or the projection of acceleration in the direction of $\vec{N}$.

$$\vec{a} = \overbrace{\underbrace{\left(\frac{d^2s}{dt^2}\right)}_{a_{\mathbf{T}}}\vec{T}}^{\text{tangential part}} + \overbrace{\underbrace{\left(\left(\frac{ds}{dt}\right)^2\kappa\right)}_{a_{\mathbf{N}}}\vec{N}}^{\text{normal part}}$$

$$\vec{a} = \overbrace{\underbrace{\left(\frac{d^2s}{dt^2}\right)}_{a_{\mathbf{T}}}\vec{T}}^{\text{tangential part}} + \overbrace{\underbrace{\left(\left(\frac{ds}{dt}\right)^2\kappa\right)}_{a_{\mathbf{N}}}\vec{N}}^{\text{normal part}}$$

We sometimes write $a_{\mathbf{T}}\vec{T} = \vec{a}_{\mathbf{T}}$ and $a_{\mathbf{N}}\vec{N} = \vec{a}_{\mathbf{N}}$.

$$\vec{a} = \underbrace{\left(\frac{d^2s}{dt^2}\right)\vec{T}}_{\vec{a}_{\mathbf{T}}\text{ tangential part}} + \underbrace{\left(\left(\frac{ds}{dt}\right)^2\kappa\right)\vec{N}}_{\vec{a}_{\mathbf{N}}\text{ normal part}}$$

The tangential part $\vec{a}_{\mathbf{T}}$ reflects changing speed, and the normal part $\vec{a}_{\mathbf{N}}$ reflects changing direction.

If an object is moving around a circle of radius R at speed V, we have $\kappa = \frac{1}{R}$, and the normal part of acceleration, also called the centripetal acceleration, has magnitude $\left(\frac{ds}{dt}\right)^2\kappa = \frac{V^2}{R}$.

If our moving object has mass m, we may divide the force $m\vec{a}$ acting on the object into two parts: $ma_{\mathbf{T}}\vec{T}$ acts to change the speed, and $ma_{\mathbf{N}}\vec{N}$ acts to change the direction of motion.

If the object is moving along a circle, we may call these the linear force and the centripetal force. The magnitude of the centripetal force is $\frac{mV^2}{R}$.

Example: Use this analysis of acceleration to find the curvature of the parabola $y = x^2$ at the point $(0,0)$.

Parametrize the parabola and compute the components of acceleration at our point:

$$\vec{r} = \left\langle t, t^2 \right\rangle \qquad \vec{v} = \langle 1, 2t \rangle \qquad \frac{ds}{dt} = |\vec{v}| = \sqrt{4t^2+1} \qquad \vec{a} = \langle 0, 2 \rangle$$

At the point $(0,0)$ we have $t = 0$, so

$$\vec{r} = \langle 0, 0 \rangle \qquad \vec{v} = \langle 1, 0 \rangle \qquad \frac{ds}{dt} = 1 \qquad \vec{a} = \langle 0, 2 \rangle$$

Now find $a_{\mathbf{N}}$, the component of acceleration normal to the direction of motion, or to $\vec{v}$. In general, we can find $a_{\mathbf{T}}\vec{T}$ as the vector projection of $\vec{a}$ in the direction of $\vec{v}$, and then set $a_{\mathbf{N}}\vec{N} = \vec{a} - a_{\mathbf{T}}\vec{T}$. In this case,

$$a_{\mathbf{T}}\vec{T} = \frac{\vec{a}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v} = \frac{0}{\vec{v}\cdot\vec{v}}\,\vec{v} = \langle 0, 0 \rangle$$

$$a_{\mathbf{N}}\vec{N} = \vec{a} - a_{\mathbf{T}}\vec{T} = \langle 0, 2 \rangle - \langle 0, 0 \rangle = \langle 0, 2 \rangle$$

$$2 = |a_{\mathbf{N}}\vec{N}| = a_{\mathbf{N}} = \left(\frac{ds}{dt}\right)^2 \kappa = (1)^2\kappa = \kappa$$

The circle of radius $\frac{1}{2}$ with center $\left(0, \frac{1}{2}\right)$ is tangent to the parabola at $(0,0)$, has the same curvature, and has the same unit normal vector. It is called the osculating circle to the parabola at $(0,0)$.

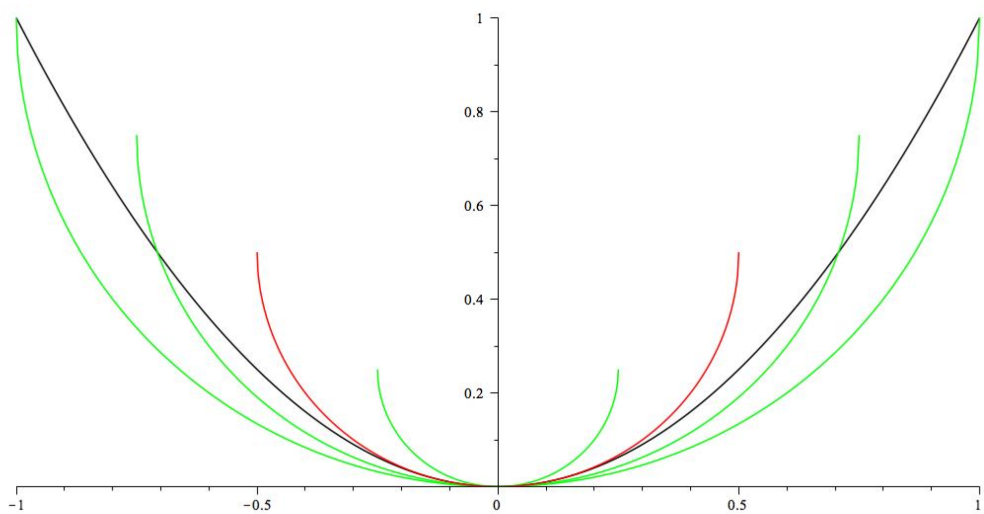

Figure 1: A black parabola and some semicircles. The red one is osculating.

Another way to derive our earlier formula for curvature:

$$\vec{a} = a_{\mathbf{T}}\vec{T} + a_{\mathbf{N}}\vec{N}$$
$$\vec{v} \times \vec{a} = \vec{v} \times \left(a_{\mathbf{T}}\vec{T} + a_{\mathbf{N}}\vec{N}\right) = \left(\vec{v} \times a_{\mathbf{T}}\vec{T}\right) + \left(\vec{v} \times a_{\mathbf{N}}\vec{N}\right)$$

When is the cross product of two vectors equal to $\vec{0}$? When then are parallel.

$$\vec{v} \times \vec{a} = \left(\vec{v} \times a_{\mathbf{N}}\vec{N}\right)$$

If two vectors are perpendicular, what is the magnitude of their cross product? The product of their magnitudes.

$$|\vec{v} \times \vec{a}| = |\vec{v}|\,|a_{\mathbf{N}}| = \left(\frac{ds}{dt}\right)\left(\frac{ds}{dt}\right)^2 \kappa = \left(\frac{ds}{dt}\right)^3 \kappa$$

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{\left(\frac{ds}{dt}\right)^3}$$

Example: Use this to find the curvature of the parabola $y = x^2$ at the point $(0, 0)$.

View the parabola as contained in the plane $z = 0$ in $\mathbb{R}^3$. Parametrize it and compute the velocity, speed, and acceleration at our point:

$$\vec{r} = \left\langle t, t^2, 0\right\rangle \qquad \vec{v} = \langle 1, 2t, 0\rangle \qquad \frac{ds}{dt} = |\vec{v}| = \sqrt{4t^2 + 1} \qquad \vec{a} = \langle 0, 2, 0\rangle$$

At the point $(0, 0, 0)$ we have $t = 0$, so

$$\vec{r} = \langle 0, 0, 0\rangle \qquad \vec{v} = \langle 1, 0, 0\rangle \qquad \frac{ds}{dt} = 1 \qquad \vec{a} = \langle 0, 2, 0\rangle$$

Now when $t = 0$ we get

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{\left(\frac{ds}{dt}\right)^3} = \frac{|\,\langle 1, 0, 0\rangle \times \langle 0, 2, 0\rangle\,|}{1} = |\,\langle 0, 0, 2\rangle\,| = 2.$$

Example: A projectile is to be fired from ground level at an angle α above the horizontal. How fast must it be fired in order for it to go 100 meters before it hits the ground? Ignore forces other than the force of gravity.

Set this problem in the xy-plane with y being vertical. The force acting on our object is the force of gravity, $\langle 0, -mg\rangle$. (All the units here are SI units.) We use Newton's second law, force equals mass times acceleration,

$$\langle 0, -mg\rangle = m\vec{a} \qquad \vec{a} = \langle 0, -g\rangle.$$

This is the acceleration of the object at all times.

Assume we fire the projectile from the origin, so initial position is $\vec{r}(0) = \langle 0, 0\rangle$. If we fire toward the positive x direction, at an angle α above the x-axis and at an initial speed of V meters per second, the initial velocity is $\vec{v}(0) = \langle V\cos\alpha, V\sin\alpha\rangle$.

Now we use these facts, and some antidifferentiation:

$$\vec{a}(t) = \langle 0, -g\rangle$$
$$\vec{v}(t) = \langle 0, -gt\rangle + \vec{C}$$
$$\vec{v}(0) = \langle 0, 0\rangle + \vec{C} = \langle V\cos\alpha, V\sin\alpha\rangle$$

We use this to find $\vec{C} = \langle V\cos\alpha, V\sin\alpha\rangle$.

$$\vec{v}(t) = \langle V\cos\alpha, V\sin\alpha - gt\rangle$$

$$\vec{r}(t) = \langle (V\cos\alpha)t, (V\sin\alpha)t - \frac{g}{2}t^2\rangle + \vec{D}$$

We use $\vec{r}(0) = \langle 0, 0\rangle$ to solve for $\vec{D}$, and we get

$$\vec{r}(t) = \langle (V\cos\alpha)t, (V\sin\alpha)t - \frac{g}{2}t^2\rangle$$

The projectile hits the ground when the y-coordinate equals 0, which it does initially (at $t = 0$), and when $V\sin\alpha = \frac{g}{2}t$, or $t = \frac{2V\sin\alpha}{g}$. The distance the projectile travels is the x-coordinate at this time, which is $(V\cos\alpha)\frac{2V\sin\alpha}{g} = \frac{2V^2\cos\alpha\sin\alpha}{g}$. For this to equal 100, we must have

$$\frac{2V^2\cos\alpha\sin\alpha}{g} = 100 \qquad V = \sqrt{\frac{50g}{\cos\alpha\sin\alpha}}.$$

Note: You can read in the textbook the proof that a projectile fired at a given speed will go farthest if the angle α is $\frac{\pi}{4}$. If we choose this angle, then our required initial speed is

$$V = \sqrt{\frac{50g}{\cos\alpha\sin\alpha}} = \sqrt{100g} = 10\sqrt{g}.$$

Exercise: Parametrize the intersection of the elliptical cone $z^2 = x^2 + 4y^2$ with the plane $z = 2$.

Write down an integral representing the arc length of this curve. (Do not try to evaluate this integral.)

If an object travels along the curve with position function given by the parametrization you chose, find the tangential and normal components of the object's acceleration, and the curvature of the curve, at the points $(2, 0, 2)$ and $(0, 1, 2)$. Use geometrical reasoning if you can.

(Hint: You're only interested in two points here, so don't do everything in full generality.)

Example: γ is the intersection of the plane $x - y = 2$ with the surface $z = x^2 + y^2$, oriented in the direction of increasing x-coordinate. (That means this is the direction of motion.) Find the unit tangent vector, unit normal vector, and curvature at any point.

First, to parametrize. Since $y = x - 2$, we have

$$z = x^2 + (x-2)^2 = x^2 + x^2 - 4x + 4 = 2x^2 - 4x + 2 + 2 = 2(x-1)^2 + 2.$$

Now we set $x = t$ (so as t increases we go in the direction of increasing x), and parametrize:

$$\vec{r} = \left\langle t, t-2, 2(t-1)^2 + 2 \right\rangle \qquad \vec{v} = \langle 1, 1, 4(t-1) \rangle \qquad \vec{a} = \langle 0, 0, 4 \rangle$$

$$\frac{ds}{dt} = |\vec{v}| = \sqrt{16(t-1)^2 + 2}$$

$$\vec{T} = \boxed{\left\langle \frac{1}{\sqrt{16(t-1)^2+2}}, \frac{1}{\sqrt{16(t-1)^2+2}}, \frac{4(t-1)}{\sqrt{16(t-1)^2+2}} \right\rangle}$$

As $\vec{a}_{\mathbf{T}}$ is the projection of $\vec{a}$ in the direction of $\vec{v}$, we have

$$\boxed{\vec{a}_{\mathbf{T}} = \frac{\vec{a} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}} = \frac{16(t-1)}{16(t-1)^2+2} \langle 1, 1, 4(t-1) \rangle = \left\langle \frac{16(t-1)}{16(t-1)^2+2}, \frac{16(t-1)}{16(t-1)^2+2}, \frac{64(t-1)^2}{16(t-1)^2+2} \right\rangle$$

$$\boxed{\vec{a}_{\mathbf{N}} = \vec{a} - \vec{a}_{\mathbf{T}}} = \left\langle -\frac{16(t-1)}{16(t-1)^2+2}, -\frac{16(t-1)}{16(t-1)^2+2}, 4 - \frac{64(t-1)^2}{16(t-1)^2+2} \right\rangle =$$

$$\left\langle -\frac{16(t-1)}{16(t-1)^2+2}, -\frac{16(t-1)}{16(t-1)^2+2}, \frac{64(t-1)^2 + 8 - 64(t-1)^2}{16(t-1)^2+2} \right\rangle =$$

$$\frac{8}{16(t-1)^2+2} \langle -2(t-1), -2(t-1), 1 \rangle$$

$\vec{N}$ is the unit vector in the direction of $\vec{a}_{\mathbf{N}}$, so

$$\vec{N} = \boxed{\frac{1}{\sqrt{8(t-1)^2+1}} \langle -2(t-1), -2(t-1), 1 \rangle}$$

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{\left(\frac{ds}{dt}\right)^3} = \frac{|\, \langle 1, 1, 4(t-1) \rangle \times \langle 0, 0, 4 \rangle \,|}{(16(t-1)^2+2)^{\frac{3}{2}}} = \frac{|\, \langle 4, -4, 0 \rangle \,|}{(16(t-1)^2+2)^{\frac{3}{2}}} = \frac{2\sqrt{2}}{(16(t-1)^2+2)^{\frac{3}{2}}}$$

We can replace t with x to find $\vec{T}$, $\vec{N}$, and κ in terms of the coordinates of a point on γ.