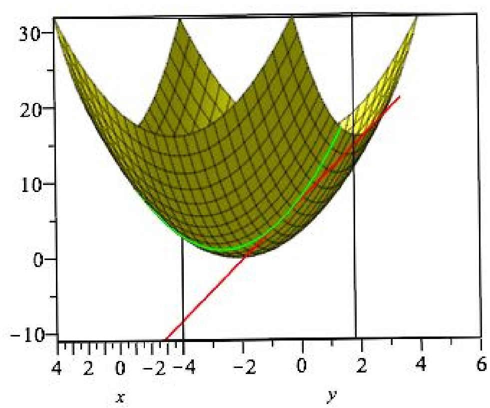
Math 8
Fall 2019
Section 2
November 6, 2019

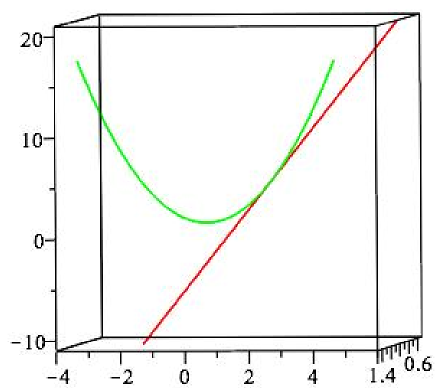
First, some important points from the last class:

Definition: The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) is the derivative of the function of x we get by setting y to have constant value y_0:

$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = D_x f(x_0, y_0) = \frac{d}{dx}\left(f(x, y_0)\right)\Big|_{x=x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}.$$

Geometrically, this is the slope (vertical rise over horizontal run, treating the z-axis as vertical) of the tangent line to the graph of f at $(x_0, y_0, f(x_0, y_0)))$ in the plane $y = y_0$.





The second partial derivatives of f include

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}.$$

Theorem (Clairaut's theorem): If suitable hypotheses hold, the corresponding mixed second partial derivatives of a function are always equal. That is,

$$f_{xy} = f_{yx} \qquad f_{xz} = f_{zx} \qquad f_{yz} = f_{zy}$$

Example: Find an equation for the tangent plane to the graph of the function

$$f(x, y) = x^2y^2$$

at the point $(1, 3, 9)$.

The partial derivatives of f at that point are

$$\frac{\partial f}{\partial x}(1,3) = (2xy^2)\Big|_{(x,y)=(1,3)} = 18$$

$$\frac{\partial f}{\partial y}(1,3) = (2x^2y)\Big|_{(x,y)=(1,3)} = 6$$

Vectors in the direction of the lines tangent to the graph of f at that point in vertical planes:

$$x = 1: \qquad \left\langle 0, 1, \frac{\partial f}{\partial y}(1,3) \right\rangle = \langle 0, 1, 6\rangle$$

$$y = 3: \qquad \left\langle 1, 0, \frac{\partial f}{\partial x}(1,3) \right\rangle = \langle 1, 0, 18\rangle$$

Vector normal to both tangent lines:

$$\langle 0, 1, 6\rangle \times \langle 1, 0, 18\rangle = \langle 18, 6, -1\rangle$$

Equation of plane containing both tangent lines (containing point $(1, 3, 9)$ and normal to the vector $\langle 18, 6, -1\rangle$):

$$\langle x - 1, y - 3, z - 9\rangle \cdot \langle 18, 6, -1\rangle = 0$$

$$18(x - 1) + 6(y - 3) - (z - 9) = 0$$

$$\underbrace{(z - 9)}_{\Delta z} = \underbrace{18}_{\frac{\partial z}{\partial x}} \underbrace{(x - 1)}_{\Delta x} + \underbrace{6}_{\frac{\partial z}{\partial y}} \underbrace{(y - 3)}_{\Delta y}$$

$$z = 18(x - 1) + 6(y - 3) + 9$$

$$z = \left(\frac{\partial f}{\partial x}(1,3)\right) \underbrace{(x - 1)}_{\Delta x} + \left(\frac{\partial f}{\partial y}(1,3)\right) \underbrace{(y - 3)}_{\Delta y} + f(1, 3)$$

Theorem: If the graph of f has a tangent plane at the point $(x_0, y_0, f(x_0, y_0))$, its equation is

$$z = \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0) + f(x_0, y_0).$$

Theorem: (the same theorem rephrased) If the graph of f has a tangent plane at the point $(x_0, y_0, f(x_0, y_0))$, it is the graph of the function

$$L(x,y) = \left(\frac{\partial f}{\partial x}(x_0,y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)(y - y_0) + f(x_0,y_0).$$

Definition: The function

$$L(x,y) = \left(\frac{\partial f}{\partial x}(x_0,y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)(y - y_0) + f(x_0,y_0)$$

is called the *linearization* of f at (x_0, y_0). We may also call it a linear approximation or a tangent approximation.

For (x, y) near (x_0, y_0), we have $f(x, y) \approx L(x, y)$. Setting $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$ (so $x - x_0 = \Delta x$ and $y - y_0 = \Delta y$), when Δx and Δy are small, we can write

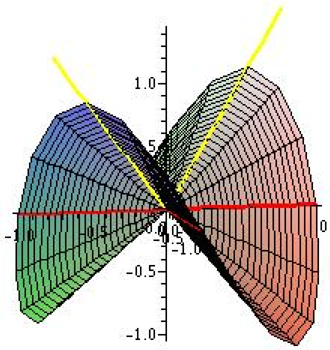
$$\begin{gathered}
f(x,y) \approx L(x,y) = \left(\frac{\partial f}{\partial x}(x_0,y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)(y - y_0) + f(x_0,y_0) \\
f(x_0 + \Delta x, y_0 + \Delta y) \approx \left(\frac{\partial f}{\partial x}(x_0,y_0)\right)(\Delta x) + \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)(\Delta y) + f(x_0,y_0). \\
f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0,y_0) \approx \left(\frac{\partial f}{\partial x}(x_0,y_0)\right)(\Delta x) + \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)(\Delta y); \\
\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y.
\end{gathered}$$

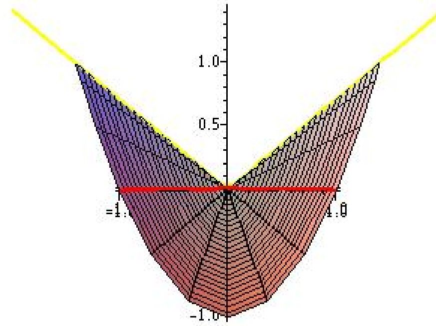
The differential is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy, \quad \text{or} \quad dz = \underbrace{\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}_{\text{One piece for each input variable!}}.$$

Everything works the same for $f : \mathbb{R}^3 \to \mathbb{R}$, or, for that matter, for $f : \mathbb{R}^n \to \mathbb{R}$.

Warning: The fact that f has partial derivatives at a point is *not enough* to guarantee that its graph has a tangent plane there. Here are two pictures of the graph of the function





$f(x,y) = \dfrac{2xy}{\sqrt{x^2+y^2}}$ (setting $f(0,0) = 0$, to make f defined and continuous everywhere). The red lines are the intersections of the graph of f with the planes $x = 0$ and $y = 0$. Both are horizontal, so $\dfrac{\partial f}{\partial x}(0,0) = \dfrac{\partial f}{\partial y}(0,0) = 0$. The yellow $\vee$ is the intersection of the graph of f with the plane $x = y$. It is pointed at the origin, and does not have a tangent line there, so the graph of f has no tangent plane at $(0,0)$.

We do, however, have this useful theorem:

Theorem: If the partial derivatives of $f(x,y)$ are defined near (x_0, y_0) and continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

Defined near (x_0, y_0) means there is some (possibly tiny) disc with center (x_0, y_0) such that the partial derivatives are defined at all points inside the disc. If this disc has radius δ, it may be called the δ-neighborhood of (x_0, y_0). Some books, instead of defined near (x_0, y_0), may say defined in some neighborhood of (x_0, y_0).

You can compute the partial derivatives of the function f pictured above, and check that although they are defined everywhere, they are not continuous at $(0,0)$.

Example: Show that

$$f(x, y, z) = xyz$$

is differentiable at the point $(1, 2, 1)$, and then use the linear approximation to f to approximate the product of the three numbers 1.01, 1.98, and .99.

The partial derivatives of f are

$$\frac{\partial f}{\partial x}(x, y, z) = yz \qquad \frac{\partial f}{\partial y}(x, y, z) = xz \qquad \frac{\partial f}{\partial z}(x, y, z) = xy.$$

They are defined and continuous everywhere (because they are polynomials), so by the theorem, f is differentiable everywhere.

For small values of Δx, Δy, and Δz, we can say

$$f(1 + \Delta x, 2 + \Delta y, 1 + \Delta z) \approx$$
$$\left(\frac{\partial f}{\partial x}(1, 2, 1)\right) \Delta x + \left(\frac{\partial f}{\partial y}(1, 2, 1)\right) \Delta y + \left(\frac{\partial f}{\partial z}(1, 2, 1)\right) \Delta z + f(1, 2, 1) =$$
$$2\Delta x + \Delta y + 2\Delta z + 2.$$

At the point $(1.01, 1.98, .99)$, we have $\Delta x = .01$, $\Delta y = -.02$ and $\Delta z = -.01$, so

$$(1.01)(1.98)(.99) = f(1.01, 1.98, .99) \approx 2(.01) + (-.02) + 2(-.01) + 2 = 1.98$$

(The actual product, per calculator, is 1.979802. Our error is .000198, which is about .01%. This seems pretty good, since Δx, Δy, and Δz were about 1% of our original numbers.)

Remember our **definition:** The function $f(x,y)$ is differentiable at the point $(x_0, y_0, f(x_0, y_0))$ if there is a function

$$L(x,y) = ax + by + c$$

(where a, b, and c are constants) such that

$$L(x_0, y_0) = f(x_0, y_0)$$

and

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - L(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0.$$

Since we know that we must have

$$L(x,y) = \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0) + f(x_0, y_0)$$

another way to phrase this definition is as follows: If

$$E(x,y) = f(x,y) - L(x,y)$$

is the error in using the linearization $L(x,y)$ of $f(x,y)$ at (x_0, y_0) to approximate f, then f is differentiable at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0.$$

Note: There is an alternative definition of differentiability in the textbook: The function $f(x,y)$ is differentiable at (x_0, y_0) if there are functions ε_1 and ε_2 such that

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_1(\Delta x, \Delta y) = 0 \quad \& \quad \lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_2(\Delta x, \Delta y) = 0$$

and we can write

$$f(x_0 + \Delta x, y_0 + \Delta y) = L(x_0 + \Delta x, y_0 + \Delta y) + (\Delta x)\varepsilon_1(\Delta x, \Delta y) + (\Delta y)\varepsilon_2(\Delta x, \Delta y),$$

where $L(x,y)$ is the linearization of $f(x,y)$ at (x_0, y_0). In other words, we have that the error $E(x_0 + \Delta x, y_0 + \Delta y)$ can be written in the form

$$(\Delta x)\varepsilon_1(\Delta x, \Delta y) + (\Delta y)\varepsilon_2(\Delta x, \Delta y)$$

where

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_1(\Delta x, \Delta y) = 0 \quad \& \quad \lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_2(\Delta x, \Delta y) = 0.$$

Both definitions are ways of saying that the error not only approaches zero as one approaches (x_0, y_0), it approaches zero very quickly. The two definitions are actually equivalent; that is, if one holds then the other one must hold as well. You can use either one.

Example: Find an equation for the tangent plane to the sphere

$$x^2 + y^2 + z^2 = 169$$

at the point $(3, 4, 12)$.

We can consider z to be a function of x and y on the top half of the sphere, so $z = f(x, y)$. The graph of the linearization of f at $(3, 4)$ will be tangent to the graph of f. We can find $\dfrac{\partial z}{\partial x}$ by implicit differentiation, treating y as a constant, z as a function of x, and x as the independent variable:

$$x^2 + y^2 + z^2 = 169$$

$$2x + 0 + 2z\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{x}{z}$$

In the same way, we get

$$\frac{\partial z}{\partial y} = -\frac{y}{z}$$

and at $(x, y, z) = (3, 4, 12)$

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = -\frac{3}{12} \qquad \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = -\frac{4}{12}$$

Our linearization (or tangent approximation) is

$$L(x, y) = \frac{\partial f}{\partial x}(3, 4)(x - 3) + \frac{\partial f}{\partial y}(3, 4)(y - 4) + f(3, 4) =$$

$$\left(\frac{-3}{12}\right)(x - 3) + \left(\frac{-4}{12}\right)(y - 4) + 12 = -\frac{x}{4} - \frac{y}{3} + \frac{169}{12}$$

so we can write our tangent plane as

$$z = -\frac{x}{4} - \frac{y}{3} + \frac{169}{12}$$

$$3x + 4y + 12z = 169.$$

Example: Use differentials to approximate the volume of metal in a cylindrical can of height 6 inches and radius 2 inches, if the top and bottom of the can are .006 inches thick, and the curved sides are .004 inches thick.

We can express volume as a function of height and radius as

$$V = \pi r^2 h.$$

We want the difference in volume between the inside and outside of the can, so we are looking for ΔV when $\Delta r = .004$ and $\Delta h = .012$, given $r \approx 2$ and $h \approx 6$. We have

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

$$dV = (2\pi r h)\, dr + \left(\pi r^2\right) dh$$

$$\Delta V \approx (2\pi r h)\, \Delta r + \left(\pi r^2\right) \Delta h.$$

Plugging everything in:

$$\Delta V \approx (2\pi(2)(6))\,(.004) + \left(2\pi(2)^2\right)(.012) = (.192)\pi$$

The can contains approximately $.192\pi$ cubic inches of metal.

Exercise: The temperature at point (x, y, z) (where distances are in meters) is given by the function $f(x, y, z) = x^2 + 2y^2 + z^4$ (in degrees Celsius).

What are the units of $\dfrac{\partial f}{\partial x}$? $\boxed{}$

Find the linearization of the function f at the point $(1, 1, 1)$.

The differential df of this function is given by

$$df = \boxed{}$$

Points P and Q are both near the point $(1, 1, 1)$, and the displacement from P to Q is $\overrightarrow{PQ} = \langle .01, .02, -.02 \rangle$. Use differentials to approximate the change in temperature when moving from point P to point Q.

Exercise: Use implicit differentiation to find the partial derivatives of z with respect to x and with respect to y on the ellipsoid

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 3$$

at the point $(3, 4, 5)$. Then find an equation for the tangent plane to the ellipsoid at that point.

Use the linear approximation to approximate the z-coordinate of the point on the ellipsoid whose x- and y-coordinates are 3.02 and 4.01.

Exercise: Show that any function of the form

$$f(x, y) = ae^{bx}\sin(by),$$

where a and b are constants, satisfies Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Exercise: Check directly that Clairaut's Theorem holds of any function of the form

$$f(x, y) = g(x)h(y),$$

where g and h are differentiable functions. (Hint: If x is constant, then $g(x)$ is also constant.)

Mathematical Challenge: Prove that the two definitions of differentiability are equivalent, by showing that if $E(x, y)$ is the error in using the linearization of $f(x, y)$ at (x_0, y_0) to approximate $f(x, y)$, and we set $\Delta x = x - x_0$ and $\Delta y = y - y_0$, then:

1. IF
$$E(x, y) = (\Delta x)\varepsilon_1(\Delta x, \Delta y) + (\Delta y)\varepsilon_2(\Delta x, \Delta y)$$
where
$$\lim_{(\Delta x, \Delta y) \to (0,0)} \varepsilon_1(\Delta x, \Delta y) = 0 \quad \& \quad \lim_{(\Delta x, \Delta y) \to (0,0)} \varepsilon_2(\Delta x, \Delta y) = 0,$$
THEN
$$\lim_{(x,y) \to (x_0, y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

2. IF
$$\lim_{(x,y) \to (x_0, y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0,$$
THEN we can write
$$E(x, y) = (\Delta x)\varepsilon_1(\Delta x, \Delta y) + (\Delta y)\varepsilon_2(\Delta x, \Delta y)$$
where
$$\lim_{(\Delta x, \Delta y) \to (0,0)} \varepsilon_1(\Delta x, \Delta y) = 0 \quad \& \quad \lim_{(\Delta x, \Delta y) \to (0,0)} \varepsilon_2(\Delta x, \Delta y) = 0.$$

Hint: Notice that $\Delta x = x - x_0$ and $\Delta y = y - y_0$. This means that $(x, y) \to (x_0, y_0)$ is the same as $(\Delta x, \Delta y) \to (0, 0)$.

Hint on part (2): You can write

$$E(x, y) = \frac{(\Delta x)^2}{(\Delta x)^2 + (\Delta y)^2} E(x, y) + \frac{(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} E(x, y).$$