Math 8
Fall 2019
Section 2
November 11, 2019

First, some important points from the last class:

Definition: If $f : \mathbb{R}^n \to \mathbb{R}$, the *gradient* of f is the vector whose components are its partial derivatives:

$$\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}(x,y,z),\ \frac{\partial f}{\partial y}(x,y,z),\ \frac{\partial f}{\partial z}(x,y,z) \right\rangle.$$

If f is differentiable, we may also call ∇f the total derivative of f and write it as f'.

Theorem (the chain rule): If $\vec{r}(t)$ is differentiable at t_0, and $f(x,y,z)$ is differentiable at $\vec{r}(t_0)$, then

$$\frac{d}{dt}\left(f(\vec{r}(t))\right) = f'(\vec{r}(t)) \cdot \vec{r}\,'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t).$$

$$\frac{dw}{dt} = \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\Delta w \approx \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z \approx \frac{\partial w}{\partial x}\frac{dx}{dt}\Delta t + \frac{\partial w}{\partial y}\frac{dy}{dt}\Delta t + \frac{\partial w}{\partial z}\frac{dz}{dt}\Delta t$$

The chain rule in different settings:

$$t \to x \to w$$
$$\frac{dw}{dt} = \frac{dw}{dx} + \frac{dx}{dt}$$

$$t \to (x,y,z) \to w$$
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$(s,t) \to (x,y,z) \to w$$
$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

Theorem (the implicit function theorem): An equation $f(x,y,z) = 0$ defining a surface S can be thought of as implicitly defining z as a function of x and y near a point on S. Then we have

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}.$$

Preliminary Homework: Let $f(x,y) = ax + by + d$, and let S be the graph of f. Note that S is a plane.

1. Two points P and Q lie on S. The coordinates of P are (x, y, z) and the coordinates of Q are $(x + \Delta x, y + \Delta y, z + \Delta z)$. Find Δz as a function of Δx and Δy.

$$\Delta z = \boxed{a\Delta x + b\Delta y}$$

2. Express $\overrightarrow{PQ}$ as the sum of two vectors, $\vec{w}_H$ horizontal (with z-component equal to 0) and $\vec{w}_V$ vertical (with x- and y-components equal to 0).

$$\vec{w}_H = \boxed{\langle \Delta x, \Delta y, 0\rangle} \qquad \vec{w}_V = \langle 0, 0, \Delta z\rangle = \boxed{\langle 0, 0, a\Delta x + b\Delta y\rangle}$$

3. If $\langle \Delta x, \Delta y\rangle = h\langle \cos\theta, \sin\theta\rangle$, find the "slope" of $\overrightarrow{PQ}$.

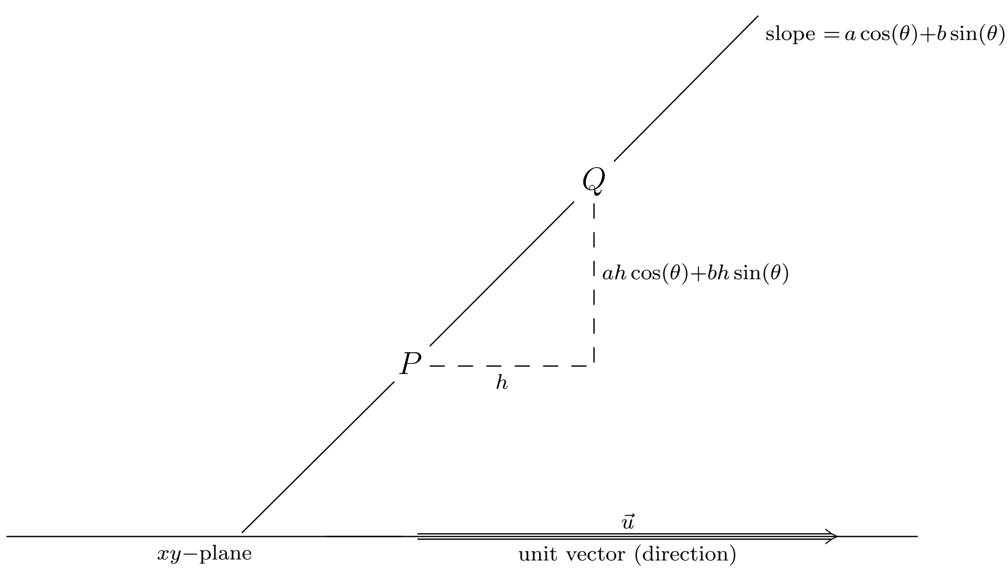
$$\text{slope} = \frac{\Delta z}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \frac{ah\cos(\theta) + bh\sin(\theta)}{\sqrt{h^2\cos^2(\theta) + h^2\sin(\theta)}} = \boxed{a\cos(\theta) + b\sin(\theta)}$$

4. This is

$$\boxed{\langle a, b\rangle \cdot \langle \cos(\theta), \sin(\theta)\rangle\,.}$$

Note $a = f_x$ and $b = f_y$ so $\langle a, b\rangle = \nabla f$.

5. Explain why this is a kind of partial derivative of f, in the direction given by the unit vector $\langle \cos(\theta), \sin(\theta)\rangle$.



Definition: If $f : \mathbb{R}^n \to \mathbb{R}$ and $\vec{u} = \langle u_1, \ldots, u_n \rangle$ is a unit vector in $\mathbb{R}^n$, then the *directional derivative* of f at $(x_1, \ldots, x_n)$ in the direction $\vec{u}$ is

$$D_{\vec{u}} f(x_1, \ldots, x_n) = \frac{\partial f}{\partial \vec{u}}(x_1, \ldots, x_n) = \lim_{h \to 0} \frac{f((x_1, \ldots, x_n) + h(u_1, \ldots, u_n)) - f(x_1, \ldots, x_n)}{h}.$$

This is the rate of change of f with respect to distance, when the argument (input) of f is moving in the direction $\vec{u}$.

If $f : \mathbb{R}^2 \to \mathbb{R}$, then $D_{\vec{u}}(x, y)$ is the slope of the slice of the graph of f in the vertical plane containing the line in the xy-plane through the point (x, y) in the direction of the vector $\vec{u}$.

Theorem: If f is differentiable at $(x_1, \ldots, x_n)$, then

$$D_{\vec{u}}(x_1, \ldots, x_n) = \nabla f(x_1, \ldots, x_n) \cdot \vec{u}.$$

Warning: The vector $\vec{u}$ must be a unit vector.

Example: For differentiable $f : \mathbb{R}^2 \to \mathbb{R}$,

$$D_{\hat{i}} f(x, y) = \nabla f(x, y) \cdot \hat{i} = \left\langle \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right\rangle \cdot \langle 1, 0 \rangle = \frac{\partial f}{\partial x}(x, y).$$

Example: Suppose $\nabla f(x, y) = \langle 3, 4 \rangle$. What is the largest possible value of $D_{\vec{u}} f(x, y)$, and for what value of $\vec{u}$ do we get this value?

$$D_{\vec{u}} f(x, y) = \nabla f(x, y) \cdot \vec{u} = |\nabla f(x, y)| \, |\vec{u}| \cos\theta = |\nabla f(x, y)| \cos\theta,$$

where θ is the angle between $\nabla f(x, y)$ and $\vec{u}$.

The maximum possible value of the directional derivative is $|\nabla f(x, y)|$, which we get when $\cos(\theta) = 1$, or $\theta = 0$, or $\vec{u}$ is in the same direction as $\nabla f(x, y)$.

In our case, the maximum possible value is $|\langle 3, 4 \rangle| = 5$, which occurs when $\vec{u}$ is in the direction of $\langle 3, 4 \rangle$, or $\vec{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.

Example: In the same situation, for what values of $\vec{u}$ is $D_{\vec{u}} f(x, y) = 0$?

When $\cos(\theta) = 0$, or $\vec{u} \perp \nabla f(x, y)$.
In our case, this happens at $\vec{u} = \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle$ and $\vec{u} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$.

Theorem: If f is differentiable at $(x_1, \ldots, x_n)$ then:

The maximum value of $D_{\vec{u}}f(x_1, \ldots, x_n)$ is $|\nabla f(x_1, \ldots, x_n)|$ and it occurs when $\vec{u}$ points in the direction of $\nabla f(x_1, \ldots, x_n)$.

The minimum value of $D_{\vec{u}}f(x_1, \ldots, x_n)$ is $-|\nabla f(x_1, \ldots, x_n)|$ and it occurs when $\vec{u}$ points in the opposite direction to $\nabla f(x_1, \ldots, x_n)$.

The value of $D_{\vec{u}}f(x_1, \ldots, x_n)$ is 0 when $\vec{u}$ is perpendicular to $\nabla f(x_1, \ldots, x_n)$.

The vector $\nabla f(x_1, \ldots, x_n)$ is normal to the level set (level curve or level surface) of f containing the point $(x_1, \ldots, x_n)$.

Example: Our famous crawling bug is on the graph of the function $f(x, y) = x^2y - xy^2 + 30$, and its shadow in the xy-plane is at the point $(1, 2)$. If the bug crawls uphill as steeply as possible, in what direction will its shadow be moving?

$$\nabla f(x, y) = \left\langle 2xy - y^2,\, x^2 - 2xy \right\rangle \qquad \nabla f(1, 2) = \langle 0,\, -3\rangle$$

Its shadow will move in the direction in which height increases fastest, or in which the directional derivative of f is greatest. This is the direction given by $\nabla f(1, 2) = \langle 0, -3\rangle$. The shadow moves in the direction of the unit vector $\langle 0, -1\rangle$.

In what direction will the bug itself be moving?

The slope of the bug's path will be the directional derivative in this direction. We chose the direction so this was the greatest possible directional derivative. This is the magnitude of the gradient, $|\nabla f(1, 2)| = |\,\langle 0, -3\rangle\,| = 3$.

A vector in the direction of the bug's motion is $\langle 0, -1, 3\rangle$ (moving a distance 1 in the direction of $\nabla f(1, 2)$ in the xy-plane, and a vertical distance of $|\nabla f(1, 2)| = 3$).

A unit vector giving the direction of the bug's motion is $\left\langle 0, \dfrac{1}{\sqrt{10}}, -\dfrac{3}{\sqrt{10}} \right\rangle$

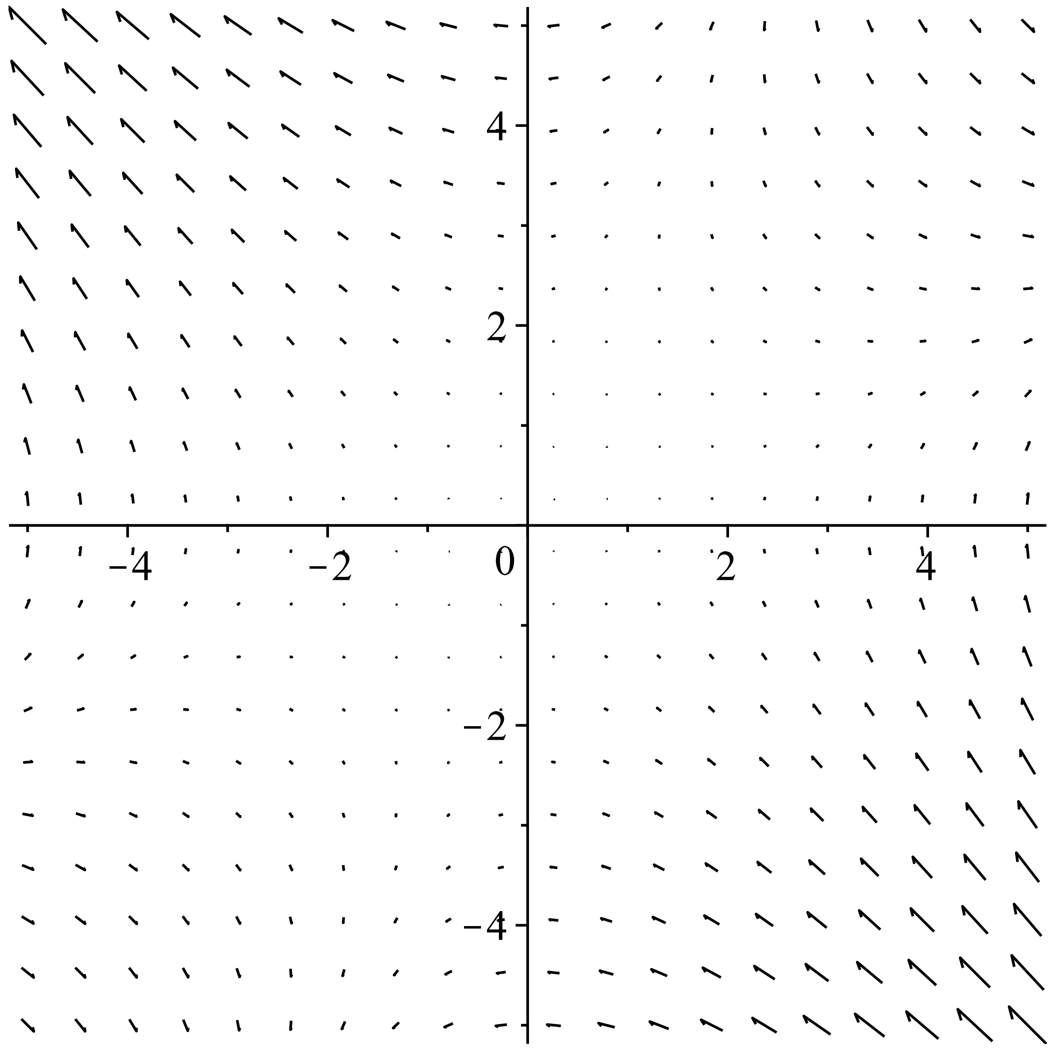
Suppose the bug's shadow is at the point $(1, 1)$ and moving toward the point $(2, 3)$. How steep is the bug's path at that point? Is the bug climbing or descending?

A vector in the direction of the shadow's motion is $\langle 1, 2\rangle$, and a unit vector in this direction is $\vec{u} = \left\langle \dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}} \right\rangle$.

The directional derivative of the function f giving the bug's height at this point, and in this direction, is

$$D_{\vec{u}}(1, 1) = \nabla f(1, 1) \cdot \vec{u} = \langle 1, -1\rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = -\frac{1}{\sqrt{5}}.$$

The slope of the bug's path is $-\dfrac{1}{\sqrt{5}}$. The bug is descending.



4
2
−2
−4
−4
−2
0
2
4

Example: Find an equation for the tangent plane to the surface $4x^2 + y^2 = z^2 + 4$ at the point $(1, 2, -2)$. Also find an equation for the normal line to this surface at this point. (This is the line, through the given point, that is normal to the surface at that point.)

We can rewrite the equation as $4x^2 + y^2 - z^2 - 4 = 0$. This is a level surface of the function $f(x, y, z) = 4x^2 + y^2 - z^2 - 4$, so a normal vector to the surface is $\nabla f(1, 2, -2)$

$$\nabla f(x, y, z) = \langle 8x, 2y, -2z\rangle \qquad \nabla f(1, 2, -2) = \langle 8, 4, 4\rangle \,.$$

This is also a normal vector to the tangent plane, and so is $\langle 2, 1, 1\rangle$. A point on the plane is $(1, 2, -2)$, so an equation for the plane is

$$2x + y + z = 2.$$

The normal line to the surface at this point has equation

$$\langle x, y, z\rangle = \langle 1, 2, -2\rangle + t\,\langle 2, 1, 1\rangle \,.$$

Example: If the temperature at point (x, y, z) is given by $f(x, y, z) = 10x + 5y^2 + z^2$, and you are located at the point $(1, 1, 3)$, in what direction should you move to cool down as quickly as possible?

The maximal directional derivative is in the direction of ∇f and has value $|\nabla f|$, and the minimal directional derivative is in the opposite direction, the direction of $-\nabla f$, and has value $-|\nabla f|$. In our example,

$$\nabla f(x, y, z) = \langle 10, 10y, 2z\rangle$$

$$-\nabla f(1, 1, 3) = -\langle 10, 10, 6\rangle = 2\langle -5, -5, -3\rangle$$

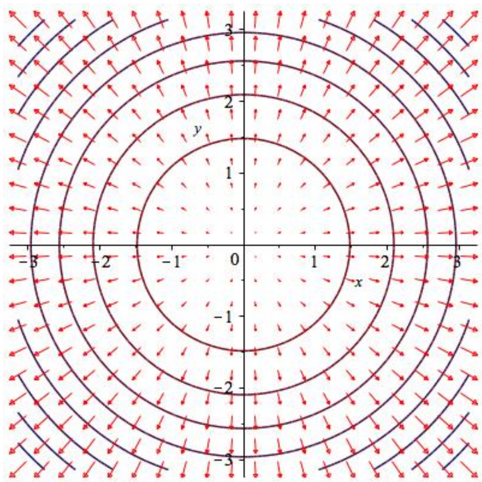
A unit vector in this direction is

$$\vec{u} = \left\langle \frac{-5}{\sqrt{59}}, \frac{-5}{\sqrt{59}}, \frac{-3}{\sqrt{59}} \right\rangle$$

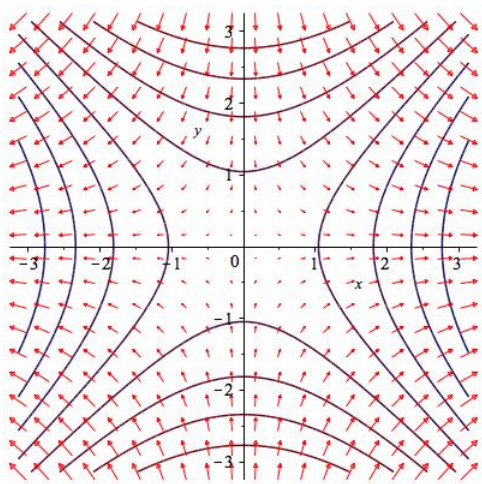
The directional derivative in this direction is $-|\nabla f(1, 1, 3)| = -|\langle 10, 10, 6\rangle| = -\sqrt{236} = -2\sqrt{59}$. We can check this by computing the directional derivative.

$$D_{\vec{u}}(1, 1, 3) = \nabla f(1, 1, 3) \cdot \vec{u} = \langle 10, 10, 6\rangle \cdot \left\langle \frac{-5}{\sqrt{59}}, \frac{-5}{\sqrt{59}}, \frac{-3}{\sqrt{59}} \right\rangle = \frac{-118}{\sqrt{59}} = \frac{-2(59)}{\sqrt{59}} = -2\sqrt{59}.$$

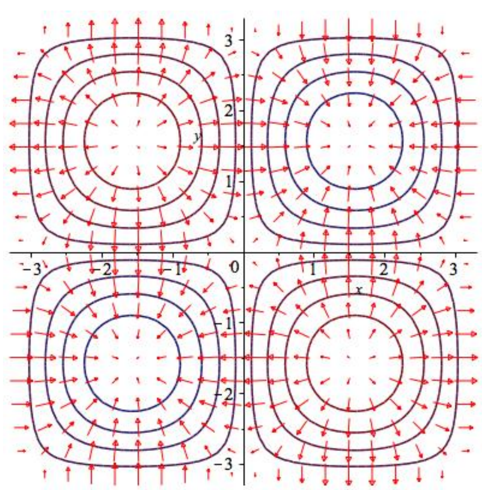
Here are some pictures of the gradient fields of functions $f(x, y)$, together with level curves of those functions. (Note that the arrows are shorter than they should be, but to scale relative to each other.)



$f(x, y) = x^2 + y^2$



$f(x, y) = x^2 - y^2$



$f(x, y) = \sin(x) \sin(y)$

Exercise: Let $f(x, y) = xe^y$ and $P = (2, 0)$. Find:

1. The directional derivative of f at P in the direction given by the vector $\langle 1, -1 \rangle$. (Remember to use a unit vector to represent the direction.)
2. The largest possible directional derivative of f at P.
3. A unit vector in the direction giving that largest possible directional derivative.

Exercise: Find equations for the tangent plane and for the normal line to the ellipsoid $4x^2 + y^2 + 9z^2 = 14$ at the point $(1, 1, 1)$.

Exercise:

$$f(x, y) = \frac{1}{x^2 + y^2 + 3}$$

1. What is the largest possible value of $D_{\vec{u}}f(3, 4)$, and for what unit vector $\vec{u}$ do we have that value?

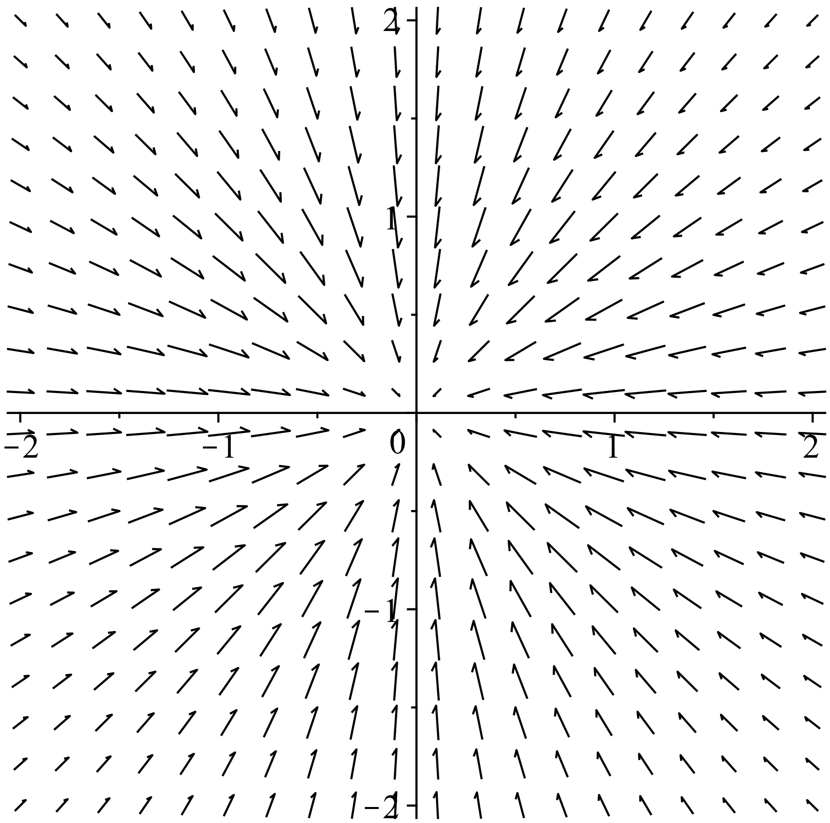
2. At any particular point (x, y), in which direction $\vec{u}$ do we get the largest possible value of $D_{\vec{u}}f(x, y)$? (Give a geometric answer.) What is that value?

3. What is the largest possible value of $D_{\vec{u}}f(x, y)$ for all possible values of $\vec{u}$, x, and y?

   (Hint: For any particular point (x, y), you just found the largest possible value of $D_{\vec{u}}f(x, y)$. That value is a function of $x^2 + y^2$. Set $t = x^2 + y^2$, and now you have a function $g(t)$ to maximize. Second hint: It is easier to maximize $(g(t))^2$.)

   At which points do we have that directional derivative?

The gradient field for this function is pictured on the next page.


2
1
0
−1
−2
−2
−1
1
2