Math 8
Fall 2019
Section 2
November 13, 2019

First, some important points from the last class:

**Definition:** If $f : \mathbb{R}^n \to \mathbb{R}$ and $\vec{u} = \langle u_1, \ldots, u_n \rangle$ is a unit vector in $\mathbb{R}^n$, then the *directional derivative* of $f$ at $(x_1, \ldots, x_n)$ in the direction $\vec{u}$ is

$$D_{\vec{u}} f(x_1, \ldots, x_n) = \frac{\partial f}{\partial \vec{u}}(x_1, \ldots, x_n) = \lim_{h \to 0} \frac{f((x_1, \ldots, x_n) + h(u_1, \ldots, u_n)) - f(x_1, \ldots, x_n)}{h}.$$

This is the rate of change of $f$ with respect to distance, when the argument (input) of $f$ is moving in the direction $\vec{u}$.

If $f : \mathbb{R}^2 \to \mathbb{R}$, then $D_{\vec{u}}(x, y)$ is the slope of the slice of the graph of $f$ in the vertical plane containing the line in the $xy$-plane through the point $(x, y)$ in the direction of the vector $\vec{u}$.

If $f$ denotes temperature in degrees, and we measure distances in meters, then the units of $D_{\vec{u}}(f)$ are degrees per meter.

**Theorem:** If $f$ is differentiable at $(x_1, \ldots, x_n)$, then

$$D_{\vec{u}}(x_1, \ldots, x_n) = \nabla f(x_1, \ldots, x_n) \cdot \vec{u}.$$

**Warning:** The vector $\vec{u}$ must be a unit vector.

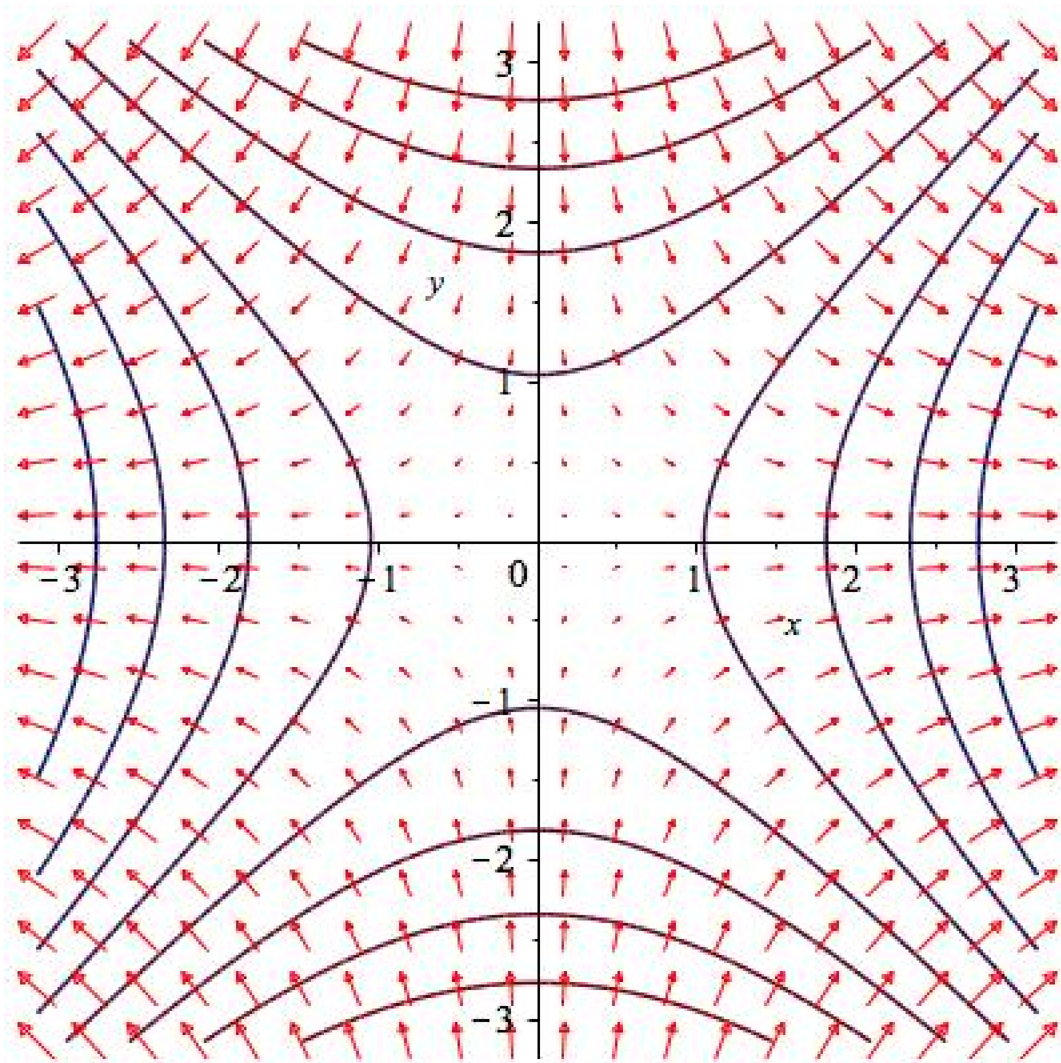
**Theorem:** If $f$ is differentiable at $(x_1, \ldots, x_n)$ then:

The maximum value of $D_{\vec{u}} f(x_1, \ldots, x_n)$ is $|\nabla f(x_1, \ldots, x_n)|$ and it occurs when $\vec{u}$ points in the direction of $\nabla f(x_1, \ldots, x_n)$.

The minimum value of $D_{\vec{u}} f(x_1, \ldots, x_n)$ is $-|\nabla f(x_1, \ldots, x_n)|$ and it occurs when $\vec{u}$ points in the opposite direction to $\nabla f(x_1, \ldots, x_n)$.

The value of $D_{\vec{u}} f(x_1, \ldots, x_n)$ is 0 when $\vec{u}$ is perpendicular to $\nabla f(x_1, \ldots, x_n)$.

The vector $\nabla f(x_1, \ldots, x_n)$ is normal to the level set (level curve or level surface) of $f$ containing the point $(x_1, \ldots, x_n)$.




Here is a contour plot, and a picture of the *gradient field*, of the function

$$f(x, y) = x^2 - y^2.$$

The gradient field is an example of a vector field, a function that assigns to every point a vector. In this case, $\nabla f$ assigns to every point $(x, y)$ the vector $\nabla f(x, y)$.

Remember that level curves of $f$ are in the domain of $f$. In this case, for $f : \mathbb{R}^2 \to \mathbb{R}$, the graph of $f$ is a surface in $\mathbb{R}^3$, but the level curves of $f$ are curves in the domain $\mathbb{R}^2$.

Gradient vectors of $f$ are also in the domain of $f$.

**Preliminary homework:** Find all points where the graph of the function

$$f(x,y) = 3x - x^3 - 2y^2 + y^4$$

has a horizontal tangent plane.

This is a polynomial, so it is differentiable everywhere, and it will have a horizontal tangent plane exactly when both partial derivatives equal zero.

$$\frac{\partial f}{\partial x}(x,y) = 3 - 3x^2 = 3(1-x^2) \qquad \frac{\partial f}{\partial y}(x,y) = -4y + 4y^3 = 4y(y^2-1)$$

$$\frac{\partial f}{\partial x}(x,y) = 0 \text{ when } x = 1 \text{ or } x = -1 \qquad \frac{\partial f}{\partial y}(x,y) = 0 \text{ when } y = 0 \text{ or } y = 1 \text{ or } y = -1$$

The tangent plane is horizontal at the points

$$(1,0) \quad (1,1) \quad (1,-1) \quad (-1,0) \quad (-1,1) \quad (-1,-1).$$

**Definition:** The point $(a, b)$ is a *critical point* of $f(x, y)$ if either $\nabla f(a, b) = \langle 0, 0\rangle$ or $\nabla f(a, b)$ is undefined.

The point $(a, b)$ is a *local maximum point* of $f(x, y)$ if there is any neighborhood of $(a, b)$ throughout which $f(x, y) \le f(a, b)$. (A neighborhood of $(x, y)$ is a disc centered at $(x, y)$.)

The point $(a, b)$ is a *local minimum point* of $f(x, y)$ if there is any neighborhood of $(a, b)$ throughout which $f(x, y) \ge f(a, b)$.

The point $(a, b)$ is a *saddle point* of $f(x, y)$ if $\nabla f(a, b) = \langle 0, 0\rangle$ and $(a, b)$ is neither a local maximum point nor a local minimum point.

**Theorem:** Local maximum and minimum points are always critical points.

**Note:** This applies to functions of more than two variables as well.

**Question:** For the function $f(x, y) = 3x - x^3 - 2y^2 + y^4$, can we tell which of the six critical points are local minimum points, local maximum points, and saddle points?

Note that

$$f(x, y) = 3x - x^3 - 2y^2 + y^4 = g(x) + h(y) \text{ where } g(x) = 3x - x^3 \text{ and } h(y) = y^4 - 2y^2.$$

We can analyze $g$ and $h$:

$$g'(x) = 3 - 3x^2 \qquad g''(x) = -6x \qquad g''(-1) = 6 > 0 \qquad g''(1) = -6 < 0,$$

so by the second derivative test, $x = -1$ is a local minimum point and $x = 1$ is a local maximum point for $g(x)$.

$$h'(y) = 4y^3 - 4y \qquad h''(y) = 12y^2 - 4 \qquad h''(-1) = h''(1) = 8 > 0 \qquad h''(0) = -4.$$

By the second derivative test, $y = -1$ and $y = 1$ are local minimum points for $h(y)$, and $y = 0$ is a local maximum point.

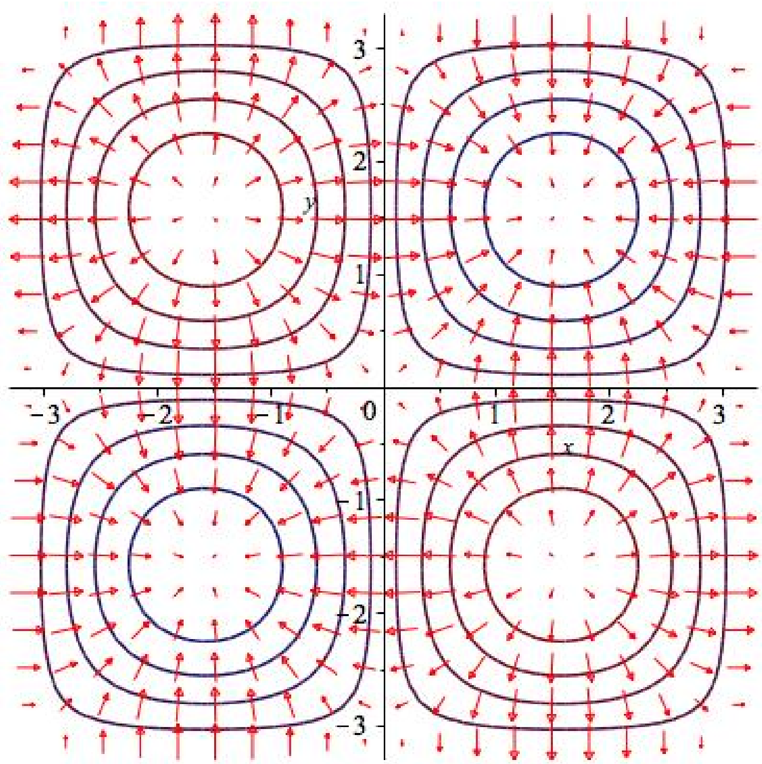
The point $(1, 0)$, where both $g(x)$ and $h(y)$ reach local maxima, is a local maximum point for $f$.

The points $(-1, -1)$ and $(-1, 1)$, where both $g(x)$ and $h(y)$ reach local minima, are local minimum points for $f$.

The points $(1, -1)$, $(1, 1)$), and $(-1, 0)$, where one of the functions reaches a local maximum and the other reaches a local minimum, are saddle points for $f$.

Usually it is not this easy, because a function $f(x, y)$ cannot usually be written in the form $g(x) + h(y)$.

**Example:** Here are some level curves, and the gradient field, of $f(x, y) = \sin(x)\sin(y)$. Where do we see critical points? Are they local maxima, local minima, or saddle points?




Some critical points are found approximately at the origin, and near the centers of the portions of the four quadrants included in this picture.

In the first and third quadrant we have local maximum points, in the second and fourth quadrant we have local minimum points, and at the origin we have a saddle point.

**Definition:** If $(a, b)$ is a critical point of $f(x, y)$, and all the second partial derivatives of $f$ are defined and continuous near $(a, b)$, we define the discriminant of $f$ at $(a, b)$ to be

$$D(a,b) = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2}(a,b) & \dfrac{\partial^2 f}{\partial x \partial y}(a,b) \\ \\ \dfrac{\partial^2 f}{\partial y \partial x}(a,b) & \dfrac{\partial^2 f}{\partial y^2}(a,b) \end{vmatrix}$$

**Theorem** (the second derivative test): If $(a, b)$ is a critical point of $f(x, y)$, and all the second partial derivatives of $f$ are defined and continuous near $(a, b)$, then

$$D(a,b) > 0 \quad \Longrightarrow \quad (a,b) \text{ is a local minimum or maximum point;}$$

$$D(a,b) < 0 \quad \Longrightarrow \quad (a,b) \text{ is a saddle point;}$$

$$D(a,b) = 0 \quad \Longrightarrow \quad \text{the second derivative test fails to give any information about (a,b).}$$

$$D(a,b) > 0 \ \ \& \ \ \frac{\partial^2 f}{\partial x^2}(a,b) < 0 \Longrightarrow (a,b) \text{ is a local maximum point;}$$

$$D(a,b) > 0 \ \ \& \ \ \frac{\partial^2 f}{\partial x^2}(a,b) > 0 \Longrightarrow (a,b) \text{ is a local minimum point.}$$

**Note:** This second derivative test is for functions $f : \mathbb{R}^2 \to \mathbb{R}$. We are not learning a second derivative test for functions $f : \mathbb{R}^3 \to \mathbb{R}$.

**Example:** Apply the second derivative test to some of the critical points of the function from the preliminary homework:

$$f(x, y) = 3x - x^3 - 2y^2 + y^4$$

$$\frac{\partial f}{\partial x}(x, y) = 3 - 3x^2 = 3(1 - x^2) \qquad \frac{\partial f}{\partial y}(x, y) = -4y + 4y^3 = 4y(y^2 - 1)$$

$$\frac{\partial^2 f}{\partial x^2}(x, y) = -6x \qquad \frac{\partial^2 f}{\partial y^2}(x, y) = 12y^2 - 4 \qquad \frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y) = 0$$

$$D(x, y) = \begin{vmatrix} -6x & 0 \\ 0 & 12y^2 - 4 \end{vmatrix} = -6x(12y^2 - 4)$$

$$D(1, 0) = -6(-4) = 24 > 0 \;\;\&\;\; \frac{\partial^2 f}{\partial x^2}(1, 0) = -6 < 0 \quad \text{so } (1, 0) \text{ is a local maximum point.}$$

$$D(1, 1) = -6(8) = -48 < 0 \quad \text{so } (1, 1) \text{ is a saddle point.}$$

**Example:** Find the critical points of

$$f(x, y) = 2x^3 - x^2 y + y$$

and use the second derivative test to classify them as local maximum points, local minimum points, or saddle points.

$$\frac{\partial f}{\partial x}(x, y) = 6x^2 - 2xy = 2x(3x - y) \qquad \frac{\partial f}{\partial y} = -x^2 + 1$$

$$\frac{\partial f}{\partial x}(x, y) = 0 \text{ when } x = 0 \text{ or } y = 3x \qquad \frac{\partial f}{\partial y}(x, y) = 0 \text{ when } x = \pm 1.$$

The critical points occur where *both* partial derivatives equal 0, at $(1, 3)$ and $(-1, -3)$.

$$D(x, y) = \begin{vmatrix} 12x - 2y & -2x \\ -2x & 0 \end{vmatrix} = -4x^2 \qquad D(1, 3) = D(-1, -3) = -4 < 0.$$

Both critical points are saddle points.

**Example:** The only critical point of $f(x,y) = x^2 + y^2$ is $(0,0)$, which is a local minimum point.

What are the largest and smallest values of $f(x,y)$ on the square region $D$ defined by $-1 \le x \le 1$, $-1 \le y \le 1$, and where are they located?

Since $f(x,y)$ is the square of the distance from the origin to $(x,y)$, we can analyze this one without too much trouble.

The smallest value is $f(0,0) = 0$.

The largest value is $f(1,1) = f(1,-1) = f(-1,1) = f(-1,-1) = 2$.

The smallest value occurs inside $D$, at a critical point of $f$. The largest value occurs at the edge of $D$. This illustrates the general case.

**Definition:** A region $D$ is *bounded* if there is some number $b$ such that every point in $D$ has a distance from the origin of at most $b$.

$D$ is *open* if every point that belongs to $D$ has a neighborhood that is included in $D$.

$D$ is *closed* if every edge point of $D$ belongs to $D$. (In three dimensions, every point on the surface of $D$ belongs to $D$.)

**Example:** The region $x^2 + y^2 < 1$ is open and bounded.

The region $x^2 + y^2 \le 1$ is closed and bounded.

The region $1 < x^2 + y^2 \le 4$ is bounded, and neither closed nor open.

The region $x^2 + y^2 \ge 1$ is closed and unbounded.

**Definition:** The number $c$ is an *absolute maximum* value for $f(x,y)$ on $D$ if there is some point $(a,b)$ in $D$ such that $f(a,b) = c$, and for all points $(x,y)$ in $D$ we have $f(x,y) \le c$. The absolute maximum value $c$ is attained at $(a,b)$.

The number $c$ is an *absolute minimum* value for $f(x,y)$ on $D$ if there is some point $(a,b)$ in $D$ such that $f(a,b) = c$, and for all points $(x,y)$ in $D$ we have $f(x,y) \ge c$. The absolute minimum value $c$ is attained at $(a,b)$.

**Theorem:** A continuous function $f(x,y)$ defined on a closed bounded region $D$ has an absolute maximum value and an absolute minimum value on $D$. The points at which those extreme values are attained are either critical points of $f$ or edge points of $D$.

**Note:** This applies to functions of more than two variables as well.

**Example:** Find the largest and smallest values of $f(x,y) = x^2 - y^2$ on the region $x^2 + y^2 \le 1$.

There is one critical point of $f$, the origin $(0,0)$, and

$$f(0,0) = 0.$$

This is a possible candidate for the largest or smallest value.

Now we have to check the edge points.

**Method 1:** Parametrize the edge, by $(x,y) = (\cos(t), \sin(t))$, $0 \le t \le 2\pi$, and find the largest and smallest values of $f(\cos(t), \sin(t))$.

$$g(t) = f(\cos(t), \sin(t)) = \cos^2(t) - \sin^2(t) = 1 - 2\sin^2(t)$$
$$g'(t) = -4\sin(t)\cos(t)$$

Check critical points of $g$ and end points of the interval. End points: $t = 0$ ($(x,y) = (1,0)$), $t = 2\pi$ ($(x,y) = (1,0)$). Critical points other than end points: When $\sin(t) = 0$, $t = \pi$, $(x,y) = (-1,0)$. When $\cos(t) = 0$, $t = \dfrac{\pi}{2}$ ($(x,y) = (0,1)$), $t = \dfrac{3\pi}{2}$ ($(x,y) = (0,-1)$). This gives these possible candidates for maximum or minimum value of $f$:

$$f(1,0) = 1 \qquad f(-1,0) = 1 \qquad f(0,1) = -1 \qquad f(0,-1) = -1.$$

Compare these to the value at our critical point, $f(0,0) = 0$.

The maximum value is $f(1,0) = f(-1,0) = 1$, and the minimum value is $f(0,1) = f(0,-1) = -1$.

**Method 2:** Write $y$ in terms of $x$ on the top half of the circle:

$$y = \sqrt{1-x^2} \qquad -1 \le x \le 1 \quad f(x,y) = f(x, \sqrt{1-x^2}) = x^2 - (1-x^2) = 2x^2 - 1 = h(x).$$

Now find the largest and smallest values of $h(x)$ by checking critical points and end points.

Critical point: $h'(x) = 4x = 0$ when $x = 0$ ($(x,y) = (0,1)$).

End points: $x = -1$ ($(x,y) = (-1,0)$), $x = 1$ ($(x,y) = (1,0)$).

This gives $(0,1)$, $(-1,0)$,and $(1,0)$ as candidate edge points at which $f$ could reach its maximum or minimum value.

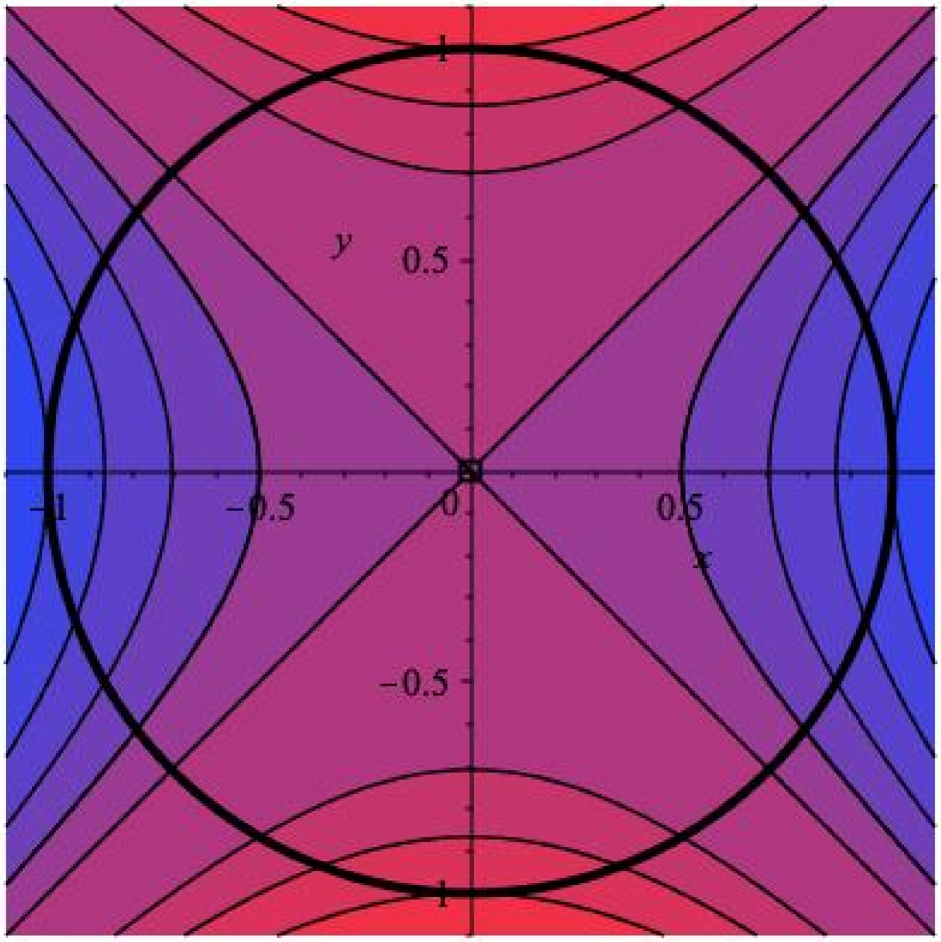
Doing the same on the bottom half of the circle gives $(0,-1)$, $(-1,0)$,and $(1,0)$.

Now we have the same five points to check as before:

$$f(1,0) = 1 \qquad f(-1,0) = 1 \qquad f(0,1) = -1 \qquad f(0,-1) = -1 \qquad f(0,0) = 0.$$

**Method 3:** We'll see another way to check edge points next class.

Here is a contour plot of the function $f(x, y) = x^2 - y^2$ from the previous example, with the edge of the region $x^2 + y^2 \leq 1$ drawn in thick black. Red regions represent lower values of $f$ and blue regions represent higher values.



**Exercise:** Find all critical points of the function

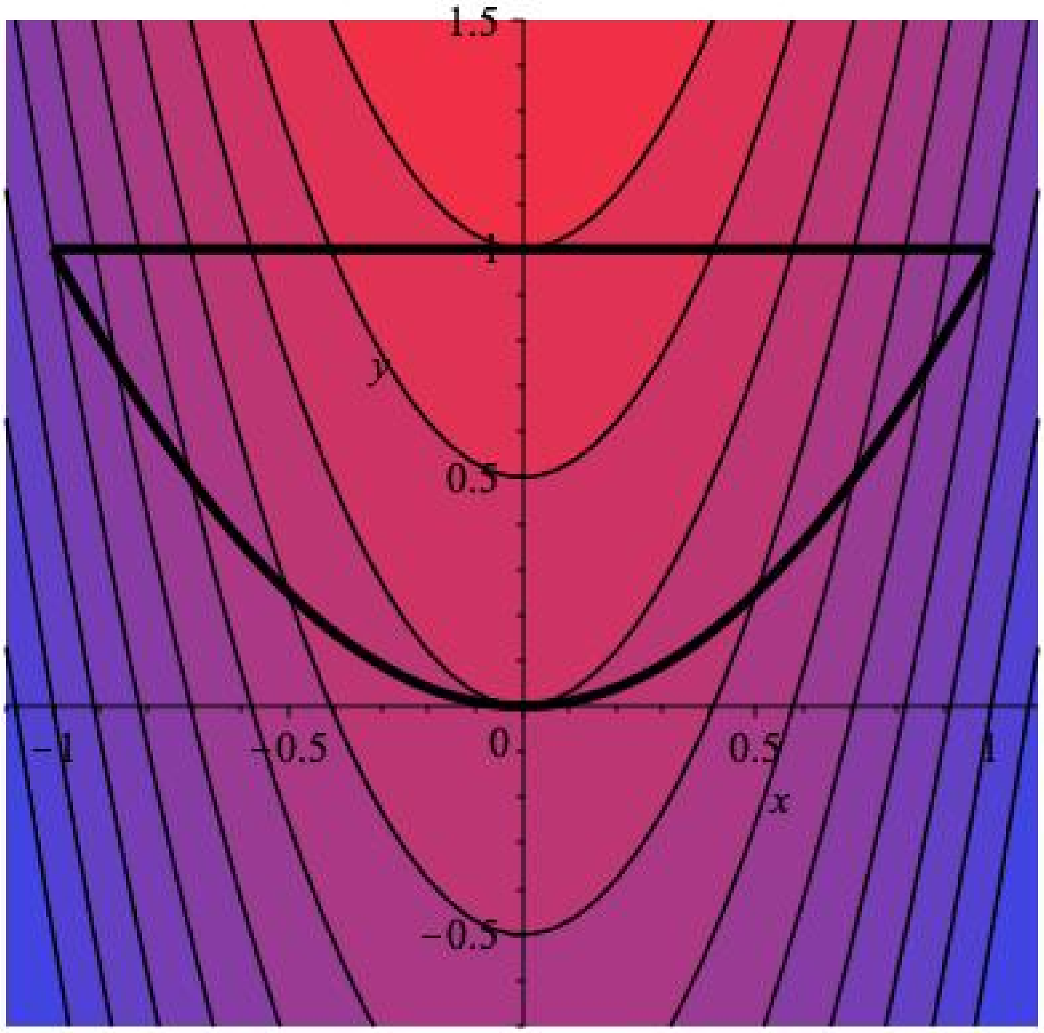
$$f(x, y) = x^3 + 3xy + y^2 + 2y$$

and classify each of them as a local maximum point, local minimum point, or saddle point.

**Exercise:** Use the second derivative test to classify the remaining critical points $(1, -1)$, $(-1, 0)$, $(-1, 1)$, $(-1, -1)$ of the function $f(x, y) = 3x - x^3 - 2y^2 + y^4$ from the preliminary homework.

**Exercise:** Find the largest and smallest values of the function $f(x, y) = 3x^2 - y$ on the region $x^2 \leq y \leq 1$, and the points at which these values are obtained. (Hint: Draw this region first.)

On the next page is a contour plot of the function $f(x, y) = 3x^2 - y$ from this exercise. Red regions represent lower values of $f$ and blue regions represent higher values. The boundary of the region $x^2 \leq y \leq 1$ is drawn in thick black lines. You should confirm that your answer to this problem agrees with the picture.


1.5
1
y
0.5
−0.5
−1
−0.5
0
0.5
1
x