Math 8
Fall 2019
Section 2
September 20, 2019

First, some important points from the last class:

$\Big(a_n\Big)_{n=1}^{\infty}$ denotes the infinite sequence $\Big(a_1, a_2, a_3, \ldots\Big)$.

$\lim\limits_{n\to\infty} a_n = L$ is defined to mean: For every $\varepsilon > 0$, there is an N, such that for all $n > N$ we have $|L - a_n| < \varepsilon$.

Important examples are sequences of finite geometric series,

$$\left(\sum_{k=0}^{n} a_0 r^k\right)_{n=0}^{\infty},$$

for which we know

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_0 r^k = \begin{cases} \dfrac{a_0}{1-r} & \text{if } |r| < 1; \\ \text{diverges} & \text{if } |r| \geq 1; \end{cases}$$

and sequences of Taylor polynomials

$$(P_n(x))_{n=0}^{\infty} = \left(\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\right)_{n=0}^{\infty}.$$

We hope to be able to show that, in many cases,

$$\lim_{n\to\infty} P_n(x) = f(x).$$

We already know this is the case when $f(x) = \dfrac{1}{1-x}$, the Taylor polynomials are centered at 0, and $|x| < 1$. This is because the Taylor polynomials are finite geometric series,

$$P_n(x) = 1 + x + x^2 + \cdots + x^n,$$

so for $|x| < 1$ we have

$$\lim_{n\to\infty} P_n(x) = \lim_{n\to\infty}(1 + x + x^2 + \cdots + x^n) = \frac{1}{1-x} = f(x).$$

However, for $|x| \geq 1$ we know $\lim\limits_{n\to\infty} P_n(x)$ diverges, so things don't always work out.

Prelmiinary Homework:

The degree 1 Taylor polynomial approximation to $f(x) = \sin x$ centered at 0 is

$$P_1(x) = x.$$

Therefore we may say that

$$\sin .01 \approx .01.$$

(Note that .01 means .01 radians.) But how close is this approximation?

First we ask how large $f(.01)$ could possibly be. We know that $f(0) = 0$ and $f'(0) = 1$. We also know that $-1 \leq f''(x) \leq 1$ for every x.

The largest possible value of $f(.01)$ for a function with these properties will occur when f grows as fast as possible between 0 and .01. That will happen when $f'(x)$ is as large as possible. The largest possible values of $f'(x)$ will occur when $f''(x)$ is as large as possible. The largest $f''(x)$ can possibly be is 1. So by assuming $f''(x) = 1$, we can find an upper bound on how large $f(.01)$ could be.

Homework:

1. Find a degree 2 polynomial $P(x)$ with the same value and derivative as $f(x) = \sin x$ at 0 and constant second derivative $P''(x) = 1$.

   How large could $\sin .01$ possibly be?

2. Find a degree 2 polynomial $P(x)$ with the same value and derivative as $f(x) = \sin x$ at 0 and constant second derivative $P''(x) = -1$.

   How small could $\sin .01$ possibly be?

3. How large could the difference $|\sin .01 - .01|$ possibly be?

In the preliminary homework for today, you answered a particular case of the following question:

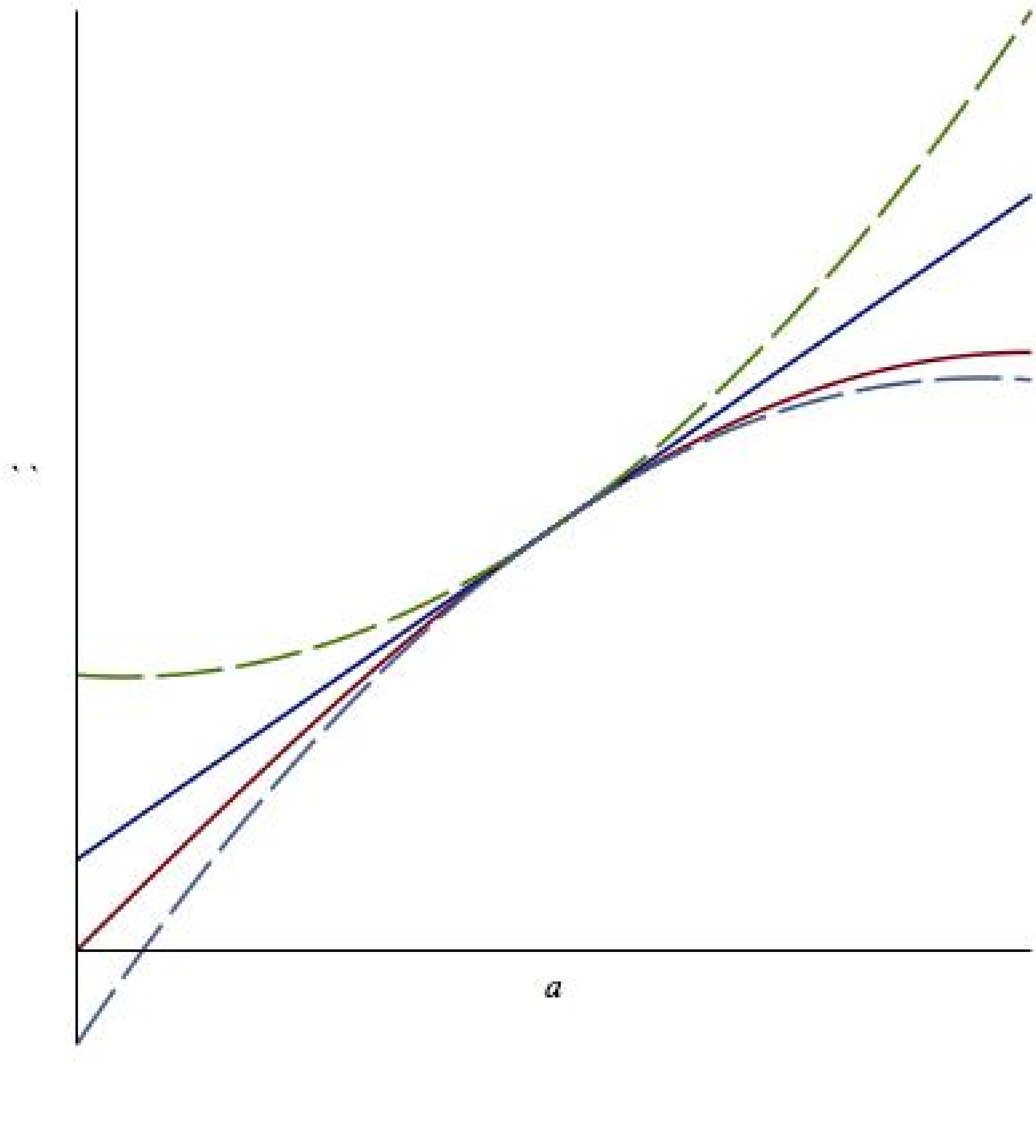
> If $P_1(x)$ is the degree 1 Taylor polynomial for $f(x)$ centered at a, then how far away from $f(x)$ could $P_1(x)$ be?

We can define $R_n(x) = f(x) - P_n(x)$. Then we are asking how large $R_1(x)$ can possibly be. Applying the same strategy you used in general, we get:

> Taylor's inequality for $n = 1$:
>
> $$|R_1(x)| \leq \frac{B_2}{2}(x-a)^2$$
>
> where B_2 is a number such that
>
> $$|f''(w)| \leq B_2$$
>
> for every w in the interval between a and x.

Note: Sometimes we define $E_n(x) = |R_n(x)| = |f(x) - P_n(x)|$. This is the error in using $P_n(x)$ as an approximation for $f(x)$. We hope $\lim_{n\to\infty} E_n(x) = 0$.

The function $f(x)$ is in red and the degree 1 Taylor polynomial centered at a is in blue. The green and gray parabolas are upper and lower bounds on $f(x)$ given by this formula.



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In these formulas, and the general formulas given below, $P_n(x)$ is the n^{th} Taylor polynomial for $f(x)$ at the point a,

$$\boxed{R_n(x) = f(x) - P_n(x)}$$

is the n^{th} remainder, $E_n(x) = |R_n(x)|$ is the n^{th} error, and the $(n+1)^{th}$ derivative of f exists everywhere between a and x.

Taylor's inequality:

$$\boxed{E_n(x) = |R_n(x)| \leq \frac{B_{n+1}}{(n+1)!}|x-a|^{n+1}}$$

where B_{n+1} is a number such that

$$|f^{(n+1)}(w)| \leq B_{n+1}$$

for every w in the interval between a and x.

Today, we'll see what Taylor's inequality can tell us.

Questions you can approach using Taylor's inequality, for a particular function $f(x)$ and center a for the Taylor polynomials, include:

1. For a particular x and n, how large can the error in using $P_n(x)$ to approximate $f(x)$ be?

2. For a particular x and error $\varepsilon > 0$, how large does n have to be in order for $P_n(x)$ to approximate $f(x)$ with an error at most ε?

3. For a particular n and error ε, for which values of x does $P_n(x)$ approximate $f(x)$ with an error at most ε?

4. For a particular x, is it the case that $\lim\limits_{n\to\infty} P_n(x) = f(x)$?

5. For which values of x is it the case that $\lim\limits_{n\to\infty} P_n(x) = f(x)$?

Example: Use the degree 5 Maclaurin polynomial for $f(x) = e^x$ to approximate e, and give a bound on the error. You can use the fact that $e < 3$.

Example: If $P_n(x)$ is the n^{th} Maclaurin polynomial for $f(x) = e^x$, then for every x,

$$\lim_{n\to\infty} P_n(x) = e^x.$$

Show this by showing that, for any particular x,

$$\lim_{n\to\infty} E_n(x) = 0.$$

$E_n(x) \leq \dfrac{B_{n+1}}{(n+1)!}|x-a|^{n+1} = \dfrac{B_{n+1}}{(n+1)!}|x|^{n+1}$, so it is enough to show

$$\lim_{n\to\infty} \frac{B_{n+1}}{(n+1)!}|x|^{n+1} = 0.$$

1. Use Taylor's inequality to find a bound on the error in using the 7^{th} Taylor polynomial for $f(x) = \cos(x)$ at the point $a = 0$ to approximate $\cos(1)$.

   Hint: When you are trying to find the number B_{n+1} in the formula, remember that the values of the sine and cosine functions are always between -1 and 1.

2. Suppose you are using $P_n(1)$ to approximate $\cos(1)$, where $P_n(x)$ is the n^{th} Taylor polynomial for $f(x) = \cos(x)$ at the point $a = 0$. Suppose you only need the error to be at most 10^{-2}. How small an n can you use?

3. Suppose you are using $P_2(x)$ to approximate $\cos(x)$, where $P_n(x)$ is the n^{th} Taylor polynomial for $f(x) = \cos(x)$ at the point $a = 0$. For what values of x are you guaranteed that the error is at most 10^{-3}?

4. Let $P_n(x)$ be the n^{th} Taylor polynomial for $f(x) = \cos(x)$ centered at the point $a = 0$. Use the definition of limit[1] to show that

$$\lim_{n \to \infty} P_n(1) = \cos(1).$$

[1] That is, suppose $\varepsilon > 0$. Find an N such that $(n > N \implies |\cos(1) - P_n(1)| < \varepsilon)$.

Now argue that, for any x, we have

$$\lim_{n\to\infty} P_n(x) = \cos(x).$$

You do not have to use the formal definition of limit.

Hint: Use Taylor's inequality to argue that

$$\lim_{n\to\infty} R_n(x) = 0.$$

5. Let $f(x) = \ln(1-x)$, and $P_n(x)$ be the n^{th} Taylor polynomial for $f(x)$ at the point $a = 0$.

   (a) Find a formula for the k^{th} derivative of $f(x)$ for $k \geq 1$.

   Write out the first few terms of $P_n(x)$ for large n. Make sure you include enough terms to see a pattern.

(b) Use Taylor's inequality to find a number n such that the n^{th} Taylor polynomial for $f(x) = \ln(1 - x)$ centered at the point $a = 0$ and evaluated at $x = .1$ approximates $\ln(.9)$ with an error of at most $.001$.

(c) For which numbers x does Taylor's inequality guarantee that you can always approximate $f(x)$ by $P_n(x)$ and make the error as small as you like by making n large enough?

(d) For which numbers x does part (a) tell you that you *cannot* approximate $f(x)$ by $P_n(x)$ and make the error as small as you like by making n large enough?

Hint: Consider $\lim_{n\to\infty} \frac{x^n}{n}$ for different values of x.

Note: Taylor's inequality may also be called Taylor's error formula.

There is a formula called the Lagrange remainder formula, from which one can deduce Taylor's inequality:

Lagrange remainder formula:

$$\boxed{R_n(x) = \frac{f^{(n+1)}(w)}{(n+1)!}(x-a)^{n+1}}$$

for some point w in the interval between a and x.

There is also an integral form of the Lagrange remainder formula:

$$\boxed{R_n(x) = \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,dt}$$

You don't have to know these. If a problem asks you to use the Lagrange remainder formula to find a bound on $E_n(x)$, you can use Taylor's inequality.