Math 8
Fall 2019
Section 2
October 4, 2019

First, some important points from the last class:

Theorem (integration by parts):

$$\boxed{\int u\,dv = uv - \int v\,du}$$

The strategy is to choose u to become simpler when differentiated, and v to become no more complicated when integrated.

Example:

$$\int \tan^{-1}(x)\,dx = \int u\,dv$$

$$u = \tan^{-1}(x) \qquad dv = dx$$

$$du = \frac{1}{x^2+1}\,dx \qquad v = x$$

$$\int u\,dv = uv - \int v\,du = x\tan^{-1}(x) - \int \frac{x}{x^2+1}\,dx =$$

$$x\tan^{-1}(x) - \frac{1}{2}\ln(x^2+1) + C.$$

Today: More examples of integrals and applications of integration, particularly work.

If a force of magnitude F acts (in the direction of motion) on an object moving a distance d, the work done by that force on that object is the product of force and distance:

$$W = Fd.$$

If the force acts opposite to the direction of motion, the work it does is negative.

If the force is variable, or not all parts of the object move the same distance, we may use an integral to find work.

In SI units we measure distance in meters, force in newtons, and work in joules.

Newtons (N) and joules (J) are derived from the basic units of distance (meter, m), mass (kilogram, kg), and time (second, s).

$$\mathrm{N} = \frac{\mathrm{kg} \cdot \mathrm{m}}{\mathrm{s}^2} \qquad \mathrm{J} = \mathrm{N} \cdot \mathrm{m}.$$

Example: Near the surface of the earth, gravity acts on an object of mass m with a force of magnitude mg directed downward. A rope of constant mass density .5 kilograms per meter and length 3 meters is hanging from the top of a wall. How much work must be done against the force of gravity to pull the rope to the top of the wall?

The quantity g is a constant, the "acceleration of gravity," approximately $9.8\frac{\mathrm{m}}{\mathrm{s}^2}$. (Here m means meters, not mass.)

(Gravity does negative work, since the motion and force have opposite directions. The magnitude is the work that must be done against the force of gravity, by some opposing force.)

(Notice that the units work out. The units of g are $\frac{\mathrm{m}}{\mathrm{s}^2}$, the units of acceleration, and the units of m are kg, the units of mass, so the units of mg are $\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}^2}$, or newtons, the units of force.)

We lay the x-axis out along the rope, with 0 at the top of the wall and 3 at the other end of the rope. Now we divide the x-axis interval $[0, 3]$ into n-many small subintervals of length Δx, and for each i, choose a point x_i in the i^{th} subinterval.

The i^{th} small piece of rope is about x_i meters from the top of the wall, so it must be lifted approximately x_i meters. It has length Δx and mass density .5, so it has mass $.5\Delta x$, and the force of gravity acting on it is $.5g\Delta x$. Therefore the work that must be done on that piece, force times distance, is (in newtons)

$$W_i \approx (.5g\Delta x)(x_i) = (.5gx_i)\Delta x.$$

We find the total work by adding up these approximations and taking a limit. The work, in newtons, is:

$$W = \lim_{n\to\infty}\left(\sum_{i=1}^{n} .5gx_i\,\Delta x\right) = \int_0^3 .5gx\,dx = (.25gx^2)\Big|_{x=0}^{x=3} = \frac{9g}{4} \approx \frac{9(9.8)}{4} = 22.05.$$

You should always include units in your answer:

$$\boxed{\text{work} \approx 22.05\,\mathrm{N}}$$

Example: The magnitude of the force of earth's gravity acting on a nearby object is $F = \dfrac{Cm}{r^2}$, where C is a constant (the product of the earth's mass and the universal gravitational constant), m is the mass of the object, and r is the distance from the object to the center of the earth.

How much work must be done against the force of gravity to launch a satellite of mass m from the surface of the earth (radius a) to height h above the earth's surface?

(Physics note: The force of gravity is not the only force acting opposite to the satellite's motion. There is also air resistance. This means the total work that must be done against the opposing forces is greater than just the work that must be done against gravity.

This is often the case with calculus work problems; we compute the work done by, or against, one particular force, which is not the only force acting. You can learn how to account for air resistance in a differential equations class.)

Draw the x-axis with $x = 0$ at the center of the earth. The satellite's path is the portion of the x-axis from $x = a$ to $x = a + h$.

Divide this path into n small intervals of length Δx, and choose a point x_i in the i^{th} subinterval. As the satellite travels along the i^{th} subinterval, the force acting on it, the distance it travels, and the work done are:

$$F_i \approx \frac{Cm}{x_i^2} \qquad \text{distance } = \Delta x \qquad W_i \approx F_i \Delta x \approx \frac{Cm}{x_i^2}\Delta x$$

Add these to approximate the total work:

$$W \approx \sum_{n=1}^{\infty} \frac{Cm}{x_i^2}\Delta x.$$

Take a limit as $n \to \infty$:

$$W = \int_a^{a+h} \frac{Cm}{x^2}\,dx = \int_a^{a+h} Cmx^{-2}\,dx =$$

$$-Cmx^{-1}\Big|_{x=a}^{x=a+h} = Cm((a)^{-1} - (a+h)^{-1}) = \frac{Cm}{a} - \frac{Cm}{a+h}$$

(Another physics note: Notice that the limit as $h \to \infty$ here is just $\dfrac{Cm}{a}$, which is finite. Since work more or less equals energy, this means that imparting a finite amount of energy to the satellite will enable it to move as far away from the earth as you like. If you simply launch the satellite up from the earth's surface (rather than carrying it up via some propulsion mechanism), the amount of energy depends on its initial velocity. "Escape velocity" is the initial velocity sufficient to guarantee it will not fall back to the earth's surface.)

(Physics meta-note: The physics notes may not be technically complete, or completely precise. The principles involved should be correct, however, even though the subtleties may be glossed over.)

Note that our work integral in the last problem was

$$\int_a^{a+h} \frac{Cm}{x^2}\,dx,$$

where $x = a$ and $x = a + h$ were the beginning and ending points of the satellite's path, and $\frac{Cm}{x^2}$ was the force acting on the satellite when it was at point x. This can work in general:

A general formula: If it makes sense (in the context of the problem) to treat an object as a single point moving along the x-axis from $x = a$ to $x = b$, acted on by a force that at point x has magnitude $F(x)$ in the x-direction (so the magnitude of the force depends only on the location x, not, for example, on time), then we can compute the work as

$$\int_a^b F(x)\,dx.$$

To see this: Break the x-axis interval $[a, b]$ into i-many subintervals of length Δx. If x_i is a point in the i^{th} subinterval, the force on the object as it travels along that subinterval is approximately $F(x_i)$, so the work done is approximately

$$W_i \approx F(x_i)\,\Delta x.$$

Add up and take a limit:

$$W = \lim_{n\to\infty}\left(\sum_{n=1}^{\infty} F(x_i)\,\Delta x\right) = \int_a^b F(x)\,dx.$$

When we cannot apply this formula:

When different parts of the object move different distances. (Example: pulling a rope to the top of a wall. We might be able to manipulate this one so we can apply the formula, but it does not naturally fit.)

When the force on the object depends on time, not just location. (Example: Launching a rocket that loses fuel, and therefore mass, as time passes. We might be able to use this formula with this problem if we could figure out how much fuel was lost when it reached height x, so we could express the mass, and therefore the force, as a function of x.)

Example: A hemispherical pool of radius 20 meters is filled with water having a mass density of $1000\dfrac{\text{kg}}{\text{m}^3}$. How much work must be done to pump it out?

Draw the x-axis from the center of the (hemi)sphere (at the surface of the water) through the bottom point of the pool, with $x = 0$ at the center of the sphere and $x = 20$ at the bottom of the tank. Divide the contents of the tank into horizontal slabs of circular cross-section. To find the radius of the horizontal cross-section at x, draw a picture of a vertical slice through the center of the pool.

We see the radius of the horizontal cross-section at x is $\sqrt{400 - x^2}$. So the cross-sectional area at x is $\pi(400 - x^2)$.

For each horizontal slab of thickness Δx, approximate the mass, the force of gravity acting on that mass, and the distance that slab must be moved to get it to the top of the pool.

If x_i is the x value at some point in the i^{th} slab, we have:
volume $\approx (\pi(400 - (x_i)^2))\Delta x$ cubic meters
mass $\approx (1000)(\pi(400 - (x_i)^2))\Delta x$ kilograms
force of gravity $\approx (1000)(\pi(400 - (x_i)^2))g\Delta x$ newtons
distance to top $\approx x_i$ meters
work to pump out $\approx (x_i)(1000)(\pi(400 - (x_i)^2))g\Delta x$ joules

The total work to pump out the pool is the sum over all the slabs (in joules):

$$W \approx \sum_{i-1}^{n} (x_i)(1000)(\pi(400 - (x_i)^2))g\Delta x.$$

Now take a limit as $n \to \infty$ to express the work (in joules) as an integral:

$$W = \int_0^{20} (x)(1000)(\pi(400 - x^2))g dx = 1000\pi g \int_0^{20} (400x - x^3) dx =$$

$$1000\pi g \left(200x^2 - \frac{x^4}{4}\right)\Bigg|_{x=0}^{x=20} = 1000\pi g(40,000) = 4\pi g \times 10^7 \approx 4(3.14)(9.8) \times 10^7 \approx 123 \times 10^9.$$

$$\boxed{\text{work} \approx 123 \times 10^9 \text{J}}.$$

Exercise: A leaky bucket is being lifted out of a well having a depth of 80 feet.

The bucket itself weighs 4 pounds, and is initially (at the bottom of the well) filled with 80 pounds of water. The water leaks out at a constant rate of .2 pounds per second, while the bucket is lifted at a constant rate of 2 feet per second.

How much work must be done to lift the bucket out of the well?

In this problem, we are using imperial units, in which the units of force are pounds and the units of work are foot-pounds. (Multiply force in pounds by distance in feet to get work in foot-pounds.)

Hint: Figure out how much the bucket weighs when it has been lifted x feet. Then you will know how much force is acting on the bucket when it is x feet above the bottom of the well.

Note: We are ignoring the weight of the rope attached to the bucket.

Exercise: The force exerted by a spring stretched x units beyond its natural length is kx, where k is a constant (called the spring constant) depending on the composition of the spring. (This is Hooke's Law, which holds as long as x is not too large.)

For example, if a spring has spring constant 200 newtons per meter, and its natural length is 5 meters, when it has been stretched to a length of 6 meters (1 meter beyond its natural length) it exerts a force of 200 newtons, and so 200 newtons of force must be applied to stretch it any further. Once it has been stretched to a length of 6.5 meters (1.5 meters beyond its natural length) it exerts a force of 300 newtons.

If the spring constant associated with a particular spring is 300 (in units of newtons per meter), how much work must be done to stretch the spring from its natural length of 20 centimeters to a length of 25 centimeters?

Suggestion: Rewrite the problem using meters instead of centimeters.
1 centimeter $= .01$ meter.

Hint: Approximately how much work is needed to stretch the spring from x meters beyond its natural length to $x + \Delta x$ meters beyond its natural length?

Exercise: A rectangular swimming pool has length 100 meters, width 60 meters, and a bottom that slopes evenly from 1 meter deep at one end to 4 meters deep at the other. It is filled with water having a mass density of $1000\frac{kg}{m^3}$. How much work must be done to pump it out?

Challenge Problem: In a fantasy novel, a magician extends a very long golden filament (of length ℓ) from the earth's surface straight into the sky. One end of the filament is just touching the ground, and the other is a distance ℓ above the earth's surface, when the spell is disrupted by a passing dragon and the gold drops back to the surface of the earth.

The magnitude of the force of earth's gravity acting on a nearby object is $F = \dfrac{Cm}{r^2}$, where C is a constant (the product of the earth's mass and the universal gravitational constant), m is the mass of the object, and r is the distance from the object to the center of the earth. The radius of the earth is a. The mass density of the filament is a constant ρ (so a piece of filament of length w has mass ρw).

Assume the filament is long enough so the force of gravity on the far end is significantly different from the force on the near end. How much work is done by the force of gravity on the falling filament?

Suggestion: Break the filament into pieces of length Δx, and use the method of the satellite problem to find the work done on each falling piece. Then add them up and take a limit.