Math 8
Fall 2019
Section 2
October 4, 2019

The leaky bucket problem:

Exercise: A leaky bucket is being lifted out of a well having a depth of 80 feet.

The bucket itself weighs 4 pounds, and is initially (at the bottom of the well) filled with 80 pounds of water. The water leaks out at a constant rate of .2 pounds per second, while the bucket is lifted at a constant rate of 2 feet per second.

How much work must be done to lift the bucket out of the well?

In this problem, we are using imperial units, in which the units of force are pounds and the units of work are foot-pounds. (Multiply force in pounds by distance in feet to get work in foot-pounds.)

Hint: Figure out how much the bucket weighs when it has been lifted x feet. Then you will know how much force is acting on the bucket when it is x feet above the bottom of the well.

Note: We are ignoring the weight of the rope attached to the bucket.

Solution 1: The bucket originally weighs 84 pounds. It is being lifted at a rate of 2 feet per second, and water is leaking out at .2 pounds per second. Therefore the water is leaking out at a rate of .2 pounds per 2 feet lifted, or .1 pound per foot lifted. Therefore, when the bucket has been lifted x feet, its weight is $84 - .1x$ pounds. The force of gravity acting on the bucket is its weight.

Now we draw the x-axis, with units of feet, extending up from the bottom of the well, with $x = 0$ at the bottom and $x = 80$ at the top. The bucket is moving from $x = 0$ to $x = 80$, and when it is at point x it has been lifted x feet so the force of gravity on the bucket is $F(x) = 84 - .1x$ pounds.

We have a problem where we have a single object moving along the x-axis from $x = 0$ to $x = 80$, and we have the force represented as a function of x, so we can integrate force with respect to distance to find work:

$$\int_0^{80} 84 - .1x\, dx = (84x - .02x^2)\Big|_{x=0}^{x=80} = 84(80) - .02(80)^2 \text{ foot-pounds}$$

Solution 2: The bucket is being lifted at a rate of 2 feet per second, so if t measures the time in seconds after it begins rising, it reaches the top of the well at time $t = 40$.

We break the time interval from $t = 0$ to $t = 40$ into n-many small intervals of length Δt seconds. If t_i is a time in the i^{th} interval, then at time t_i the bucket has been rising for t_i-many seconds, so $.2t_i$ pounds of water have leaked out (because water leaks out at a rate of .2 pounds per secone), and its weight is $84 - .2t_i$. Over that time interval of time Δt the bucket rises a distance of $2\Delta t$ feet (because it rises at a rate of 2 feet per second). Therefore the work done on the bucket over the i^{th} time interval is approximately $(84 - .2t_i)(2\Delta t)$ foot-pounds.

We approximate the work as a Riemann sum

$$W \approx \sum_{i=1}^{n} (84 - .2t_i)(2\Delta t)$$

and take a limit as $n \to \infty$ to get an integral

$$W = \int_0^{40} (84 - .2t)2\, dt \text{ foot-pounds.}$$

Relating These Solutions: Since the bucket begins at distance $x = 0$ at time $t = 0$ and rises at a rate of 2 feet per second to distance $x = 80$ at time $t = 40$, we can relate x to t:

$$x = 2t \qquad t = \frac{x}{2}.$$

If you take the integral from solution 2, and use $x = 2t$, $dx = 2dt$, to make a direct substitution, you get the integral from solution 1. On the other hand, if you take the integral from solution 1, and use $t = \frac{x}{2}$, $dt = \frac{1}{2}\,dx$ to make a direct substitution, you get the integral from solution 1.

You can do something like this in general. If an object moves from point $x = a$ to point $x = b$ between times $t = c$ and $t = d$ at velocity $\frac{dx}{dt}$, during which time it is acted on by a variable force F, you can compute the work done by F as

$$\int_a^b F\,dx$$

or as

$$\int_c^d F\frac{dx}{dt}\,dt.$$

Which you choose will depend on whether you can most easily express F as a function of time or of distance.

You do not have to remember this second formula (for computing work by integrating over time). You can always arrive at it the way we did in the second solution to the bucket problem, by approximating the work by a Riemann sum and thus arriving at an integral.