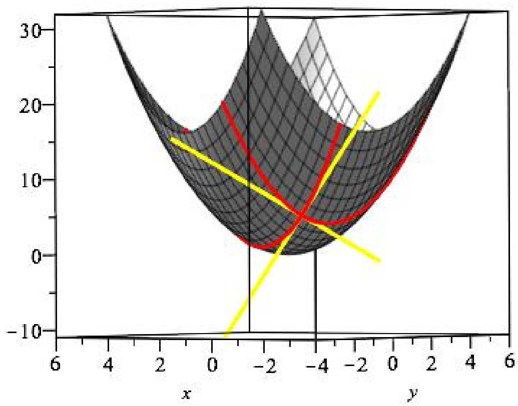


Figure 1: Trying to find the slope of a graph

Math 8
Fall 2019
Tangent Approximations and Differentiability

We have seen how to find derivatives of functions from $\mathbb{R}$ to $\mathbb{R}$ and to $\mathbb{R}^n$. What should the derivative of a function from $\mathbb{R}^n$ to $\mathbb{R}$ be?

Our first answer might be that the derivative should be the slope. However, it's not that simple.

Figure 1 shows the graph of the function $f(x, y) = x^2 + y^2$. We can try to find the slope of this graph at some point by looking at a path on the graph going through that point and finding the slope of the path; that is, by finding the slope of the tangent line to the path, as (vertical) rise over (horizontal) run. The two red curves in Figure 1 are paths going through the same point, and the yellow lines are the tangent lines to those curves at that point. One line is steeper than the other. The graph has different slopes in different directions.

Consider the plane $z = x$. The y-axis ($z = 0$, $x = 0$), which is horizontal, lies in that plane. So does the line $z = x$, $y = 0$, which makes a 45 degree angle with the xy-plane. In one direction, the slope is 0; in another, it is 1.

We could decide to take the slope in whatever direction is steepest. In the case of the plane $z = x$, then, we would say the slope is 1. However, that is not a very satisfying solution. The plane $z = y$ also has slope 1 in its steepest direction. However, the two planes $z = x$ and $z = y$ do not slant in the same direction. We would like graphs passing through the same point with the same derivative to be tangent to each other.

We want the derivative of a function to tell us about the slope of its graph in every possible direction.

For functions from $\mathbb{R}$ to $\mathbb{R}$, the derivative gives us a tangent line. For functions from $\mathbb{R}^2$ to $\mathbb{R}$, the derivative should give us a tangent plane. Looking at Figure 1, it seems that the plane containing the two yellow lines should be tangent to the graph of f, and, in fact, it is.

A (non-vertical) line in $\mathbb{R}^2$ is the graph of a function $\ell(x) = ax + b$; the slope is determined by a. If this line is tangent to the graph of f at some point, then the derivative of f at that x-value is $f'(x) = a$.

In the same way, a (non-vertical) plane in $\mathbb{R}^3$ is the graph of a function $\mathcal{P}(x, y) = ax + by + d$; the slope in every direction is determined by a and b. If this plane is tangent to the graph of f at some point, we will say the derivative of f at that (x, y)-value is $f'(x, y) = \langle a, b\rangle$.

To finish the story, we need to answer three questions.

> What does it mean for a plane to be tangent to the graph of a function?
>
> If the graph of f does have a tangent plane at a given point, how can we find it?
>
> What about derivatives of functions from $\mathbb{R}^3$ to $\mathbb{R}$? Or functions from $\mathbb{R}^2$ to $\mathbb{R}^2$? (A useful example: $f(x, y) = (r, \theta)$, where r and θ are the polar coordinates of the point with rectangular coordinates (x, y).) Or, in general, functions from $\mathbb{R}^m$ to $\mathbb{R}^n$?

In this handout, we will answer the first question and, to a certain extent, the third question. We will see how to answer the second question very soon.

The intuition from Figure 1 might suggest that a plane containing two different lines tangent to the graph of f at some point must be tangent to

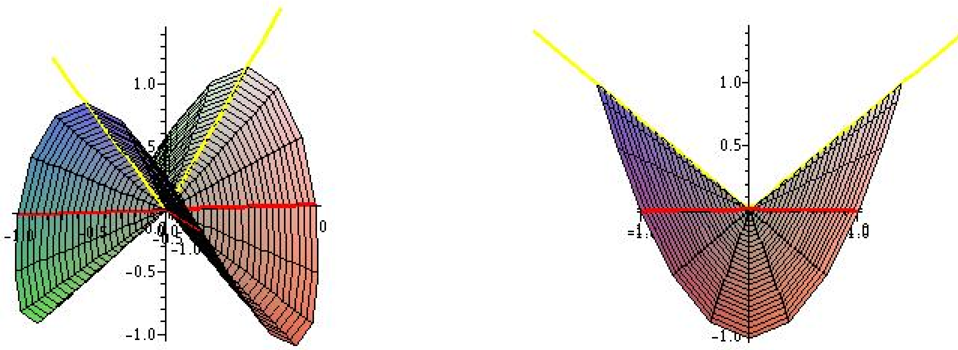


Figure 2: Graph of $f(x, y) = \dfrac{2xy}{\sqrt{x^2 + y^2}}$.
If $(x, y) = (r \sin\theta, r\cos\theta)$, then $f(x, y) = r\sin(2\theta)$.

the graph of f at that point. However, that is not always the case. If the graph of f has any tangent plane at all, it must be the plane containing those two lines. However, it is possible for a graph to have tangent lines in two different directions without having a tangent plane.

Here is an example. Figure 2 shows two different pictures of a portion of the graph of the function

$$f(x, y) = \frac{2xy}{\sqrt{x^2 + y^2}}.$$

The x- and y-axes are drawn in red, and the intersection of the graph with the vertical plane $y = x$ is drawn in yellow.

The x- and y-axes lie on the graph of f. Therefore, if we were using the "plane containing two tangent lines" definition, we would conclude that the horizontal plane $z = 0$ is tangent to the graph of f at the origin. However, as is most easily seen in the second picture, when we slice in the plane $x = y$, the graph of f has a sharp point at the origin, like the absolute value function. Therefore, it cannot have a tangent plane.

So what does it mean for the plane

$$z = ax + by + d$$

to be tangent to the surface

$$z = f(x, y)$$

at some point (x_0, y_0, z_0) that is on both the plane and the surface?

We answer this questions for functions from $\mathbb{R}$ to $\mathbb{R}$ by defining the slope of the graph to be the limit of slopes of secant lines, and then saying a line is tangent to the graph of they have the same slope. More precisely, to find the slope of the graph of f at $(x_0, f(x_0))$, we look at slopes of secant lines with one endpoint $(x_0, f(x_0))$ and the other endpoint $(x, f(x))$, and take the limit as $x \to x_0$:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

We will do something like this, but it's not quite this simple. We have already seen that the slopes of a surface in different directions are likely to be different. We can define the slope of a secant line with endpoints $(x_0, y_0, f(x_0, y_0))$ and $(x, y, f(x, y))$ as (vertical) rise over (horizontal) run, where the run is the distance in the xy-plane between the points (x_0, y_0) and (x, y).

$$slope = \frac{f(x, y) - f(x_0, y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = \frac{f(x, y) - f(x_0, y_0)}{|(x, y) - (x_0, y_0)|}.$$

But then the limit

$$\lim_{(x,y) \to (x_0, y_0)} slope = \lim_{(x,y) \to (x_0, y_0)} \frac{f(x, y) - f(x_0, y_0)}{|(x, y) - (x_0, y_0)|},$$

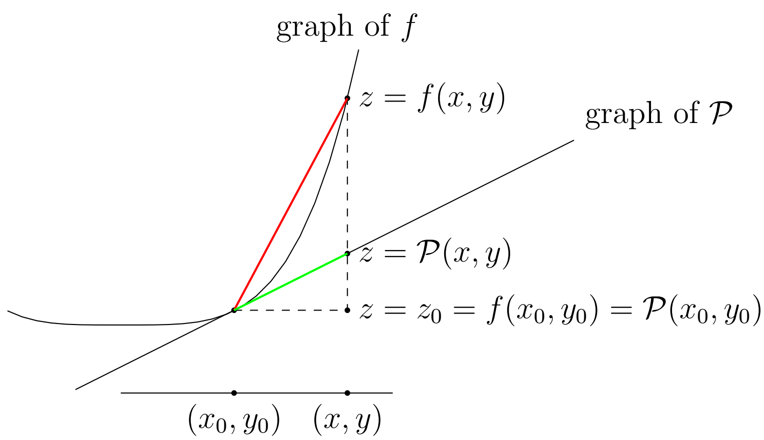
probably doesn't exist, even if the graph of f *is* a plane.

Try it yourself, with the example $f(x, y) = x$ and $(x_0, y_0) = (0, 0)$. You can check that the limit

$$\lim_{(x,y) \to (x_0, y_0)} \frac{f(x, y) - f(x_0, y_0)}{|(x, y) - (x_0, y_0)|} = \lim_{(x,y) \to (0,0)} \frac{f(x, y) - f(0, 0)}{|(x, y) - (0, 0)|} = \lim_{(x,y) \to (0,0)} \frac{x}{\sqrt{x^2 + y^2}}$$

does not exist.

However, we still want to use this idea. If a plane is tangent to the graph of f at some point, then in the limit, secant lines of the graph of f and line segments in the plane should have the same slope. We will say this by saying that, in the limit, the difference of the slopes is zero.



Suppose that the point (x_0, y_0, z_0) lies on both the graph of f and the plane

$$z = \mathcal{P}(x, y) = ax + by + d.$$

This means we must have

$$f(x_0, y_0) = \mathcal{P}(x_0, y_0) = z_0.$$

Taking a point (x, y) close to (x_0, y_0) in the domain of f gives us (red) secant line of the graph of f, with endpoints $(x_0, y_0, f(x_0, y_0))$, $(x, y, f(x, y))$ and slope

$$\frac{f((x, y) - f(x_0, y_0)}{|(x, y) - (x_0, y_0)|} = \frac{f((x, y) - z_0}{|(x, y) - (x_0, y_0)|},$$

and it also gives us a (green) line segment in the plane, with endpoints $(x_0, y_0, \mathcal{P}(x_0, y_0)$, $(x, y, \mathcal{P}(x, y))$ and slope

$$\frac{\mathcal{P}(x, y) - \mathcal{P}(x_0, y_0)}{|(x, y) - (x_0, y_0)|} = \frac{\mathcal{P}(x, y) - z_0}{|(x, y) - (x_0, y_0)|}.$$

The difference of these slopes is

$$\frac{f(x, y) - z_0}{|(x, y) - (x_0, y_0)|} - \frac{\mathcal{P}(x, y) - z_0}{|(x, y) - (x_0, y_0)|} = \frac{f(x, y) - \mathcal{P}((x, y)}{|(x, y) - (x_0, y_0)|}.$$

We say that the plane is tangent to the graph of f at (x_0, y_0, z_0) if this difference of slopes approaches 0 as (x, y) approaches (x_0, y_0):

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \mathcal{P}(x,y)}{|(x,y) - (x_0,y_0)|} = 0.$$

Definition: If $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}$ have $\boxed{f(x_0, y_0) = \mathcal{P}(x_0, y_0) = z_0}$, then the graphs of f and $\mathcal{P}$ are tangent at (x_0, y_0, z_0) if and only if

$$\boxed{\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \mathcal{P}(x,y)}{|(x,y) - (x_0,y_0)|} = 0.}$$

Definition: The function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (x_0, y_0) if its graph has a non-vertical tangent plane at the point $(x_0, y_0, f(x_0, y_0))$. That is, the graph of some function

$$\boxed{\mathcal{P}(x, y) = ax + by + d = \langle a, b\rangle \cdot \langle x, y\rangle + d}$$

is tangent to the graph of f where $(x, y) = (x_0, y_0)$. Then the derivative of f at (x_0, y_0) is

$$\boxed{f'(x_0, y_0) = \langle a, b\rangle\ .}$$

Putting these together:

$$f'(x_0, y_0) = \langle a, b\rangle$$

if and only if there is a constant d such that the graph of the function

$$\mathcal{P}(x, y) = \langle a, b\rangle \cdot \langle x, y\rangle + d$$

is tangent to the graph of f at (x_0, y_0), which means

$$\mathcal{P}(x_0, y_0) = f(x_0, y_0) \qquad \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \mathcal{P}(x,y)}{|(x,y) - (x_0,y_0)|} = 0.$$

We use this same definition for functions $f : \mathbb{R}^3 \to \mathbb{R}$:

$$f'(x_0, y_0, z_0) = \langle a, b, c \rangle$$

if and only if there is a constant d such that the graph of the function

$$\mathcal{T}(x, y, z) = ax + by + cz + d = \langle a, b, c \rangle \cdot \langle x, y, z \rangle + d$$

is tangent to the graph of f at (x_0, y_0, z_0), which means

$$\mathcal{T}(x_0, y_0, z_0) = f(x_0, y_0, z_0) \qquad \lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{f(x, y, z) - \mathcal{T}(x, y, z)}{|(x, y, z) - (x_0, y_0, z_0)|} = 0.$$

We use the same definition for $f : \mathbb{R}^n \to \mathbb{R}$:

$$f'(x_0, y_0, z_0, \dots) = \langle a, b, c, \dots \rangle$$

if and only if there is a constant d such that the graph of the function

$$\mathcal{T}(x, y, z, \dots) = ax + by + cz + \cdots + d = \langle a, b, c, \dots \rangle \cdot \langle x, y, z, \dots \rangle + d$$

is tangent to the graph of f at $(x_0, y_0, z_0, \dots)$, which means

$$\mathcal{T}(x_0, y_0, z_0, \dots) = f(x_0, y_0, z_0, \dots)$$

$$\lim_{(x,y,z,\dots)\to(x_0,y_0,z_0,\dots)} \frac{f(x, y, z, \dots) - \mathcal{T}(x, y, z, \dots)}{|(x, y, z, \dots) - (x_0, y_0, z_0, \dots)|} = 0.$$

For $F : \mathbb{R}^m \to \mathbb{R}^n$, we can set

$$F(x_1, x_2, \dots, x_m) = (F_1(x_1, \dots, x_m),\ F_2(x_1, \dots, x_m),\ \dots,\ F_n(x_1, \dots, x_m)).$$

Then we say F is differentiable when $F_1, \dots, F_n$ all are, and we define the derivative of F coordinatewise:

$$F'(p_1, p_2, \dots, p_m) =$$

$$(F_1'(p_1, p_2, \dots, p_m),\ F_2'(p_1, p_2, \dots, p_m),\ \dots,\ F_n'(p_1, p_2, \dots, p_m)).$$

(This is not quite right. This is a vector of vectors, which actually should be arranged into a matrix.)

Example: Show that the graphs of the functions $f(x,y) = x^2 - y^2$ and $\mathcal{P}(x,y) = 2x - 4y + 3$ are tangent at the point $(1, 2, -3)$.

What does this say about the derivative of f?

First we check that $(1, 2, -3)$ is on both graphs:

$$f(1,2) = (1)^2 - (2)^2 = 1 - 4 = -3$$

$$\mathcal{P}(1,2) = 2(1) - 4(2) + 3 = 2 - 8 + 3 = -3.$$

Now we use the definition of tangent. Here our point is $(x_0, y_0) = (1, 2)$, so we need to check that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \mathcal{P}(x,y)}{|(x,y) - (x_0,y_0)|} = \lim_{(x,y)\to(1,2)} \frac{f(x,y) - \mathcal{P}(x,y)}{|(x,y) - (1,2)|} = 0.$$

Evaluating the expression inside the limit:

$$\frac{f(x,y) - \mathcal{P}(x,y)}{|(x,y) - (1,2)|} = \frac{(x^2 - y^2) - (2x - 4y + 3)}{\sqrt{(x-1)^2 + (y-2)^2}} = \frac{x^2 - 2x + 1 - y^2 + 4y - 4}{\sqrt{(x-1)^2 + (y-2)^2}}$$

$$= \frac{(x-1)^2 - (y-2)^2}{\sqrt{(x-1)^2 + (y-2)^2}} = \frac{(x-1)^2}{\sqrt{(x-1)^2 + (y-2)^2}} - \frac{(y-2)^2}{\sqrt{(x-1)^2 + (y-2)^2}}$$

Looking at the first piece of this:

$$\frac{(x-1)^2}{\sqrt{(x-1)^2 + (y-2)^2}} = \frac{\sqrt{(x-1)^4}}{\sqrt{(x-1)^2 + (y-2)^2}} = \sqrt{\frac{(x-1)^4}{(x-1)^2 + (y-2)^2}} =$$

$$\sqrt{(x-1)^2 \left(\frac{(x-1)^2}{(x-1)^2 + (y-2)^2}\right)} \leq \sqrt{(x-1)^2} = |x-1|.$$

(The $\leq$ step is because the numerator of $\left(\frac{(x-1)^2}{(x-1)^2 + (y-2)^2}\right)$ is less than or equal to the denominator, so the fraction is at most 1.)

In the same way, we have

$$\frac{(y-2)^2}{\sqrt{(x-1)^2 + (y-2)^2}} \leq |y-2|,$$

and so

$$\left|\frac{f(x,y)-\mathcal{P}(x,y)}{|(x,y)-(1,2)|}\right| = \left|\frac{(x-1)^2}{\sqrt{(x-1)^2+(y-2)^2}} - \frac{(y-2)^2}{\sqrt{(x-1)^2+(y-2)^2}}\right|$$

$$\leq \left|\frac{(x-1)^2}{\sqrt{(x-1)^2+(y-2)^2}}\right| + \left|\frac{(y-2)^2}{\sqrt{(x-1)^2+(y-2)^2}}\right| \leq |x-1|+|y-2|.$$

Now

$$\lim_{(x,y)\to(1,2)} |x-1| = 0 \qquad \text{and} \quad \lim_{(x,y)\to(1,2)} |y-2| = 0,$$

and therefore

$$\lim_{(x,y)\to(1,2)} \left|\frac{f(x,y)-\mathcal{P}(x,y)}{|(x,y)-(1,2)|}\right| = 0,$$

and so,

$$\lim_{(x,y)\to(1,2)} \frac{f(x,y)-\mathcal{P}(x,y)}{|(x,y)-(1,2)|} = 0.$$

This is what we needed to show.

What does this say about the derivative of f?

Since the graph of $\mathcal{P}$ is a plane tangent to the graph of f at the point where $(x,y)=(1,2)$, this says that f is differentiable at $(1,2)$.

Furthermore, we can write

$$\mathcal{P}(x,y) = \langle 2,-4\rangle \cdot \langle x,y\rangle + 3,$$

and therefore

$$f'(1,2) = \langle 2,-4\rangle.$$

We have not yet discussed how to find the function $\mathcal{P}(x,y)$ to begin with. We will do that in a later class.

Pretty much all our derivatives and tangent approximations can be put in the same basket:

For $f : \mathbb{R} \to \mathbb{R}$, the tangent approximation at $x = x_0$ is given by a function $\mathcal{T}(x) = ax + d$, and $f'(x_0) = a$.

$(\mathcal{T}(x) = f'(x_0)(x - x_0) + f(x_0) = \underbrace{\boxed{f'(x_0)}}_{a}\, x + \boxed{(-x_0 f'(x_0) + f(x_0))})$

For $\vec{r} : \mathbb{R} \to \mathbb{R}^n$, the tangent approximation at $t = t_0$ is given by a function $\vec{\mathcal{T}}(t) = t\vec{a} + \vec{d}$, and $\vec{r}\,'(t_0) = \vec{a}$.

$(\vec{\mathcal{T}}(t) = (t - t_0)\vec{r}\,'(t_0) + \vec{r}(t_0) = \underbrace{\boxed{\vec{r}\,'(t_0)}}_{\vec{a}}\, t + \boxed{(-t_0\vec{r}\,'(t_0) + \vec{r}(t_0))})$

For $f : \mathbb{R}^n \to \mathbb{R}$ the tangent approximation at $(x, y, z \dots) = (x_0, y_0, z_0 \dots)$ is given by a function $\mathcal{T}(x, y, z, \dots) = \underbrace{\langle a, b, c, \dots \rangle}_{\vec{a}} \cdot \langle x, y, z, \dots \rangle + d$, and $f'(x_0, y_0, z_0 \dots) = \langle a, b, c, \dots \rangle$.

In fact, to complete the parallel, we will have $\mathcal{T}(x, y, z, \dots) = f'(x_0, y_0, z_0 \dots) \cdot \langle x - x_0, y - y_0, z - z_0, \dots \rangle + f(x_0, y_0, z_0 \dots)$.

In other words: Near a given point, approximate the function f by a function $\mathcal{T}$ that equals a constant multiple of the input (input times A), plus a constant D. If the graphs of f and $\mathcal{T}$ are tangent, then A is the derivative of f at that point.

We can define tangent in the same way in all case:

$$\mathcal{T}(x_0) = f(x_0) \qquad \lim_{x \to x_0} \frac{f(x) - \mathcal{T}(x)}{|x - x_0|} = 0,$$

where x, $f(x)$, 0, etc. may be scalars or vectors, depending on the domain and range of f.

Constant multiple refers to ordinary multiplication, scalar multiplication, or dot product, depending on what kind of function we are talking about.

In fact, there is a way to express all of these as the same kind of multiplication, namely matrix multiplication. For $F : \mathbb{R}^m \to \mathbb{R}^n$, this gives a way to express the tangent approximation to F as a constant multiple of the input plus a constant. We will not get into that in Math 8, however.

Note: Our definition of differentiable is different from the textbook's, and that is because our definition of tangent is different. (The two definitions are equivalent, they are just phrased differently.)

Suppose we define $E(x,y) = f(x,y) - \mathcal{P}(x,y)$ to be the error in using $\mathcal{P}(x,y)$ as an approximation to $f(x,y)$ near (x_0,y_0). Then our definition of tangent (assuming that $f(x_0,y_0) = \mathcal{P}(x_0,y_0)$) is that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{|(x,y)-(x_0,y_0)|} = 0.$$

The textbook's definition says we can write the error as the sum of two parts

$$E(x,y) = \varepsilon_1(x,y)(x-x_0) + \varepsilon_2(x,y)(y-y_0)$$

in such a way that

$$\lim_{(x,y)\to(x_0,y_0)} \varepsilon_1(x,y) = 0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} \varepsilon_2(x,y) = 0.$$

As we said, these two definitions are equivalent, and you may use either one.

In the example on page 8, we had $(x_0,y_0) = (1,2)$, and our error function $E(x,y) = f(x,y) - \mathcal{P}(x,y)$ was computed to be $(x-1)^2 - (y-2)^2$. We can write this as

$$E(x,y) = \underbrace{(x-1)}_{\varepsilon_1(x,y)}\underbrace{(x-1)}_{(x-x_0)} + \underbrace{(-(y-2))}_{\varepsilon_2(x,y)}\underbrace{(y-2)}_{(y-y_0)},$$

and check that

$$\lim_{(x,y)\to(1,2)} (x-1) = 0 \quad \text{and} \quad \lim_{(x,y)\to(1,2)} (-(y-2)) = 0.$$

This worked out nicely for this example. However, suppose we want to show that the graph of $\mathcal{P}(x,y) = x+y$ is tangent to the graph of $f(x,y) = \sin(x+y)$ at $(0,0)$. We can find

$$E(x,y) = f(x,y) - \mathcal{P}(x,y) = \sin(x+y) - x - y.$$

It's hard to see how we should rewrite this in the form $\varepsilon_1(x,y)(x-0) + \varepsilon_2(x,y)(y-0)$. It's also hard to see how to show

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)-x-y}{|(x,y)-(0,0)|} = 0,$$

but we do know what we are trying to do.