Math 8
Fall 2019
Taylor Polynomials and Taylor Series Day 4

Infinite Series

Our major interest in discussing limits of sequences is to find limits of Taylor polynomials,

$$\lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

We may write this limit instead as

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Definition: An infinite series is a sum of infinitely many terms,

$$\sum_{k=0}^{\infty} a_k.$$

(A sequence is a list; a series is a sum.) The sum is defined as follows: The n^{th} partial sum is the sum of the terms out to a_n,

$$S_n = \sum_{k=0}^{n} a_k,$$

and the sum of the series is the limit of the partial sums,

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=0}^{n} a_k.$$

If this limit exists and is a number, the series converges; if not, it diverges.

Definition: The Taylor series for $f(x)$ centered at a is the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

This is sort of the degree infinity Taylor polynomial. Its n^{th} partial sum is the degree n Taylor polynomial for $f(x)$ centered at a.

Remark: We hope that the sum of the Taylor series for $f(x)$ is equal to $f(x)$. In many nice cases, it is. Taylor's inequality (Taylor's error formula) can help us figure out when that happens.

Geometric Series

Definition: The series $\sum_{n=0}^{\infty} a_n$ is a *geometric series* with ratio r if for every n we have $\frac{a_{n+1}}{a_n} = r$; in other words, $a_{n+1} = a_n r$.

If $\sum_{n=0}^{\infty} a_n$ is a geometric series with ratio r, we can write

$$a_1 = a_0 r \qquad a_2 = a_1 r = (a_0 r)r = a_0 r^2 \qquad a_3 = a_2 r = (a_0 r^2)r = a_0 r^3 \qquad \cdots \qquad a_n = a_0 r^n.$$

Therefore, we can write the series as

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_0 r^n = a_0 \sum_{n=0}^{\infty} r^n.$$

But $\sum_{n=0}^{\infty} x^n$ is just the Taylor series centered at 0 for the function $f(x) = \frac{1}{1-x}$, and we have already seen that

$$\lim_{n\to\infty} \sum_{k=0}^{n} x^k = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \begin{cases} \frac{1}{1-x} & \text{if } |x| < 1; \\ \text{divergent} & \text{if } |x| \geq 1. \end{cases}$$

This gives us the following proposition.

Proposition:
If $\sum_{n=0}^{\infty} a_n$ is a geometric series with ratio r and first term a_0, then

$$\sum_{n=0}^{\infty} a_n = \begin{cases} \frac{a_0}{1-r} & \text{if } |r| < 1; \\ \text{divergent} & \text{if } |r| \geq 1. \end{cases}$$

Some Rules for Series

There is an extensive theory of sequences and series, most of which we will not see in Math 8. In this section, we state a few rules that should make sense. This is not a collection of facts to memorize. This is reassurance that your common sense conclusions about series and sequences are generally valid.

Generally the limits A and B in these rules are assumed to be numbers. The rules also apply to limits of ∞ and $-\infty$, as long as the expression you are evaluating is defined ($\infty + \infty = \infty$) rather than undefined ($\infty - \infty$ is undefined). Be warned that the quotient $\frac{\infty}{0}$ is undefined, *not* ∞. That is because if a_n approaches ∞ and b_n approaches 0 while oscillating between positive and negative values, then $\frac{a_n}{b_n}$ will also oscillate between positive and negative values, and therefore will not approach ∞.

You are free to use these rules in any homework or exam problem (unless the instructions say otherwise, such as, "use the definition of limit.") You do not have to cite the rule by name, as long as you make clear what fact you are using.

1. (constant multiple rule)

   If c is a constant, then

$$\left(\sum_{n\to\infty} a_n = A\right) \implies \sum_{n\to\infty} (ca_n) = cA.$$

2. (addition and subtraction rules)

$$\left(\sum_{n\to\infty} a_n = A \;\&\; \sum_{n\to\infty} b_n = B\right) \implies \sum_{n\to\infty} (a_n \pm b_n) = A \pm B.$$

3. (tail end rule)

$$\sum_{n=0}^{\infty} a_n \text{ converges } \iff \sum_{n=k}^{\infty} a_n \text{ converges.}$$

   In fact,

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n.$$

4. (comparison rule)

   If $a_n \le b_n$ for all n, then

$$\left(\sum_{n\to\infty} a_n = A \;\&\; \sum_{n\to\infty} b_n = B\right) \implies A \le B.$$

5. (decreasing terms rule)

   If $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{n\to\infty} (a_k) = 0$.

   The converse of this is false, as you can see from the series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots,$$

   which does not converge even though the individual terms do approach zero.

6. (nonnegative series rule)

   If $a_n \geq 0$ for all n, then $\sum_{n=0}^{\infty} a_n$ either converges to a finite sum or approaches $+\infty$.

These rules basically follow from applying sequence rules to the sequences of partial sums. For example, the nonnegative series rule follows from the monotone sequence theorem, since if $a_n \geq 0$ for all n, then the sequence of partial sums is an increasing sequence.

Convergence and Power Series

The Ratio Test

You may notice that we have listed many more sequence rules than series rules. This is because there are many more elementary methods for finding the limit of a sequence than for finding the sum of a series.

Sometimes, even if we cannot find the sum of a series, we can determine whether the series converges or not. There are a number of different convergence tests for series. We will make use of one particular test, the ratio test, which is very useful when we are considering Taylor series.

In order to understand why the ratio test works, we will also consider some other facts about convergent series. Be warned, the next proposition applies *only* to nonnegative series, which are series with no negative terms.

Proposition: (the comparison test) If $0 \leq b_n \leq a_n$ for all n, then

$$\sum_{n=0}^{\infty} a_n \text{ converges} \implies \sum_{n=0}^{\infty} b_n \text{ converges.}$$

This proposition follows from the fact that a series with nonnegative terms must either converge or approach infinity. If the larger series does not approach infinity, the smaller one cannot do so either, so it must converge.

The following definition turns out to be useful.

Definition: The series $\sum_{n=0}^{\infty} a_n$ is *absolutely convergent* if the series $\sum_{n=0}^{\infty} |a_n|$ is convergent.

The reason it is useful is the following proposition.

Proposition: If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then it is convergent.

This proposition follows from things we have already noted. By the sum rule, we can break up a series into the sum of its positive terms and the sum of its negative terms, and show each of those series converges by comparison to the sum of the absolute values of the terms. (There are more details in the last section.)

Be warned that the converse of this proposition is false. There are some series that are convergent but not absolutely convergent. The alternating harmonic series,

$$\sum_{n=1}^{\infty}\left((-1)^{n+1}\left(\frac{1}{n}\right)\right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots ,$$

is one important example.

We can use this proposition to prove $\sum_{n=0}^{\infty} a_n$ converges by proving $\sum_{n=0}^{\infty} |a_n|$ converges. This is often easier, because we have tests such as the comparison test that apply only to nonnegative series.

We are now set up to state and prove the ratio test:

Proposition: (the ratio test for nonnegative series) If $a_n \geq 0$ for all n, and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L,$$

then

$$\sum_{n=0}^{\infty} a_n \text{ is } \begin{cases} \text{convergent} & \text{if } L < 1; \\ \text{divergent} & \text{if } L > 1; \\ \text{we cannot tell from this test} & \text{if } L = 1. \end{cases}$$

We can rephrase this using our earlier proposition that absolutely convergent sequences are always convergent.

Proposition: (the ratio test) For any series, if

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L,$$

then

$$\sum_{n=0}^{\infty} a_n \text{ is } \begin{cases} \text{absolutely convergent} & \text{if } L < 1; \\ \text{divergent} & \text{if } L > 1; \\ \text{we cannot tell from this test} & \text{if } L = 1. \end{cases}$$

There is a proof of the ratio test in the last section. Intuitively, if we have a nonnegative series with $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}$, then a tail end of the series behaves very much like a geometric series with ratio $\frac{1}{2}$, which converges. On the other hand, if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$, then eventually the terms of the series are getting larger and larger, and the sum of larger and larger numbers must approach $+\infty$.

Radius of Convergence

Taylor series centered at $x = a$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

are examples of power series centered at $x = a$

$$\sum_{k=0}^{\infty} c_k(x-a)^k \qquad \text{(each } c_k \text{ is a constant).}$$

We can use a power series to define a function,

$$g(x) = \sum_{k=0}^{\infty} c_k(x-a)^k,$$

whose domain is the set of x for which the power series converges.

We hope that the Taylor series for $f(x)$ not only converges, but converges to $f(x)$.

To find the set of x for which a given power series converges, a good place to start is the ratio test. For example, consider

$$\sum_{k=0}^{\infty} k(x-1)^k.$$

To see whether this series converges for a particular value of x, we use the ratio test:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)(x-1)^{k+1}}{k(x-1)^k}\right| = \lim_{k\to\infty}\left|\frac{k+1}{k}(x-1)\right| = \left(\lim_{k\to\infty}\left|\frac{k+1}{k}\right|\right)|x-1| = |x-1|.$$

By the ratio test, this power series converges absolutely if $|x-1| < 1$ and diverges if $|x-1| > 1$; the ratio test doesn't tell us what happens for $|x-1| = 1$. Therefore the domain of the function defined by this power series contains $(0, 2)$, and may or may not contain the points 0 and 2.

To see whether it contains the endpoints of the interval, we can plug them in and see what happens. For $x = 2$, we have

$$\sum_{k=0}^{\infty} k(x-1)^k = \sum_{k=0}^{\infty} k(1)^k = 0 + 1 + 2 + 3 + \cdots$$

and for $x = 0$ we have

$$\sum_{k=0}^{\infty} k(x-1)^k = \sum_{k=0}^{\infty} k(-1)^k = 0 - 1 + 2 - 3 + \cdots$$

both of which diverge. Therefore the domain of this function is $(0, 2)$.

In general, we can apply the ratio test to the power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ and get

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{c_{k+1}(x-a)^{k+1}}{c_k(x-a)^k}\right| = \lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}(x-a)\right| = \left(\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|\right)|x-a|.$$

If $\left(\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|\right) = Q$, then we get

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = Q|x-a|,$$

and the power series converges absolutely when

$$Q|x-a| < 1, \text{ or } |x-a| < \frac{1}{Q},$$

and diverges if

$$Q|x-a| > 1, \text{ or } |x-a| > \frac{1}{Q}.$$

We call $R = \frac{1}{Q}$ the *radius of convergence.* (If $Q = 0$ the radius of convergence is $+\infty$, and if $Q = +\infty$ the radius of convergence is 0.) The power series converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$. For $|x-a| = R$, it may converge or diverge, depending on the power series.

It turns out that power series always behave this way, even if $\left(\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|\right)$ does not converge.

Proposition 0.1. *A power series* $\sum_{k=0}^{\infty} c_k(x-a)^k$ *always has a radius of convergence* R *with* $0 \le R \le \infty$. *The power series converges absolutely for* $|x-a| < R$ *and diverges for* $|x-a| > R$.

If $R = 0$ *the power series converges only for* $x = a$ *and if* $R = \infty$ *it converges for all* x. *For* $0 < R < \infty$, *the set of* x *for which the power series converges is one of the intervals*

$$(a-R, a+R),\ [a-R, a+R),\ (a-R, a+R],\ \text{or } [a-R, a+R].$$

This is called the interval of convergence of the series.

This proposition tells us nothing about what function the power series converges to.

Sums of Taylor Series

There are at least three methods to determine what a Taylor series converges to.

Special Examples

We might recognize a series we know about, or a series we can analyze in some clever way. For example, the Maclaurin series for $f(x) = \dfrac{1}{x^2+1}$ is

$$1 - x^2 + x^4 - x^6 + \cdots .$$

This is a geometric series with first term 1 and ratio $r = -x^2$, so we know it converges to $\dfrac{1}{1-(-x^2)} = \dfrac{1}{x^2+1} = f(x)$ for $|-x^2| < 1$ (that is, for $|x| < 1$) and diverges for $|x| \geq 1$. For $x = 1$ or $x = -1$ we can substitute 1 or -1 for x, and see that the series we get $(1 - 1 + 1 - 1 + 1 \cdots)$ diverges.

Taylor's Inequality

In our study of Taylor polynomials, we have shown that if $P_n(x)$ is the n^{th} degree Taylor polynomial for $f(x)$ centered at $x = a$, and B_{n+1} is a bound on $|f^{(n+1)}(u)|$ for u between x and a, then

$$|f(x) - P_n(x)| \leq \frac{B_{n+1}}{(n+1)!}|x-a|^{n+1}.$$

If we can show that

$$\lim_{n\to\infty} \frac{B_{n+1}}{(n+1)!}|x-a|^{n+1} = 0,$$

then we will have shown that

$$\lim_{n\to\infty} |f(x) - P_n(x)| = 0,$$

or

$$\lim_{n\to\infty} P_n(x) = f(x).$$

For example, suppose $f(x) = \sin(x)$. Then the n^{th} derivative of $f(x)$ is either $\pm\sin(x)$ or $\pm\cos(x)$, and has absolute value at most 1. Therefore, we have $B_{n+1} = 1$ for every n, and

$$\lim_{n\to\infty} \frac{B_{n+1}}{(n+1)!}|x-a|^{n+1} = \lim_{n\to\infty} \frac{|x-a|^{n+1}}{(n+1)!} = 0.$$

To see this, note that for any particular x the quantity $|x-a|$ is a constant, and we have already seen that $\lim\limits_{n\to\infty} \dfrac{c^n}{n!} = 0$.

Therefore, for every value of x, the Taylor series for $\sin(x)$ centered at a converges to the value $\sin(x)$. In particular, for $a = 0$ we get the Maclaurin series for $\sin(x)$, so we can say

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x^{2k+1}}{k!}\right) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

This means that for any x we can approximate $\sin(x)$ as closely as we like by using a Maclaurin polynomial of degree n if n is large enough. We can also go back to Taylor's inequality to see how large n has to be, for any given value of x and desired level of accuracy.

Integrals and Derivatives

We will see this method next time.

Proof That the Ratio Test Works: To prove the ratio test proposition, we first prove that absolutely convergent series are convergent.

Suppose that $\sum_{n=0}^{\infty} |a_n|$ converges. We define two new series, one including all the positive terms of our original series, and the other including all the (absolute values of) negative terms:

$$b_n = \begin{cases} a_n & \text{if } a_n \geq 0; \\ 0 & \text{if } a_n < 0. \end{cases} \qquad c_n = \begin{cases} 0 & \text{if } a_n \geq 0; \\ -a_n & \text{if } a_n < 0. \end{cases}$$

It is not hard to check that for every n we have:

$$(1.)\ 0 \leq b_n \leq |a_n| \qquad (2.)\ 0 \leq c_n \leq |a_n| \qquad (3.)\ a_n = b_n - c_n \qquad (4.)\ |a_n| = b_n + c_n.$$

From (1.) we see that $\sum_{n=0}^{\infty} b_n$ is a positive series, and since $\sum_{n=0}^{\infty} |a_n|$ converges, it follows from the comparison test that $\sum_{n=0}^{\infty} b_n$ also converges. Let its sum be B.

From (2.) we see that $\sum_{n=0}^{\infty} c_n$ is a positive series, and since $\sum_{n=0}^{\infty} |a_n|$ converges, it follows from the comparison test that $\sum_{n=0}^{\infty} c_n$ also converges. Let its sum be C.

From (3.) and the subtraction rule for series, we see that

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (b_n - c_n) = B - C.$$

In particular, $\sum_{n=0}^{\infty} a_n$ converges.

Now we prove the ratio test. Suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

First, suppose that $L > 1$, and choose $\varepsilon > 0$ small enough that $L - \varepsilon > 1$. By the definition of limit, there is some N such that whenever $n > N$ we have $\left|\frac{a_{n+1}}{a_n}\right| - L < \varepsilon$. This means that

$$\left|\frac{a_{n+1}}{a_n}\right| > L - \varepsilon > 1,$$

and that means that $|a_{n+1}| > |a_n|$. Since the individual terms of the series are getting larger in absolute value, they are not approaching zero, and so the series must diverge.

Now, suppose that $L < 1$, and choose $\varepsilon > 0$ small enough that $L + \varepsilon < 1$. By the definition of limit, there is some N such that whenever $n > N$ we have $\left|\frac{a_{n+1}}{a_n}\right| - L < \varepsilon$. This means that

$$\left|\frac{a_{n+1}}{a_n}\right| < L + \varepsilon.$$

Consider the series,

$$\sum_{n=N+1}^{\infty} |a_n|.$$

For this series, we have

$$\frac{|a_{n+1}|}{|a_n|} < L + \varepsilon,$$
$$|a_{n+1}| \leq (L + \varepsilon)|a_n|,$$

which means the terms of the series are less than or equal to the terms of a geometric series with first term a_{N+1} and ratio $L + \varepsilon$. Since $L + \varepsilon < 1$, this geometric series converges, and therefore the smaller series $\sum_{n=N+1}^{\infty} |a_{n+1}|$ also converges.

Since a tail end of the series $\sum_{n=0}^{\infty} |a_n|$ converges, the series itself converges. This means the original series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, and therefore it is convergent.