Practice Set for Exam 1, Math 9

1. Let
$$a_n = \frac{\left((-1)^{n+1} - cos(n\pi)\right)}{n^2}$$
What is the limit of the sequence $\{a_n\}$? Justify your answer.

2. Let $f(x) = (1+x)^{2001}$. What is the coefficient in the Maclaurin Series (i.e. Taylor series at 0) of $f(x)$
   (a) that stands in front of x^0;
   (b) that stands in front of x^{2001};
   (c) that stands in front of x^{2002}.

   Give a justification for the above answers.

3. Use the fact that the Maclaurin series for e^x is
$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$
to find the Maclaurin series for $\frac{e^x + e^{-x}}{2}$.

4. (a) Consider a 300 gallon tank which is filled with 100 gallons of fresh water. Suppose that a water and salt solution is poured into the tank at a rate of 10 gallons/min and that the concentration of the salt in this solution is 0.5 lbs/gal. Find the amount of salt (in pounds) that is in the tank just before the tank begins to overflow.
   (b) Assume now that, just before the tank begins to overflow, we stop pouring and we allow the well mixed solution to drain at a rate of 1 gallon/min. Describe now the amount of salt in the tank at any time t before the tank is emptied.

5. Solve the following differential equation.
$$y' + \frac{2t}{t^2+10} y = t.$$

6. (a) Solve the differential equation $y' = 3y^2 t$ subject to the condition $y(0) = 0$. (You may want to use the fact that if a solution exists then it is unique.)

   (b) Solve the same differential equation subject to the condition $y(0) = 1$.

7. Consider the third-order linear differential equation $P(x)\, y''' + Q(x)\, y'' + R(x)\, y' + S(x) y = 0$, where $y = y(x)$ is the unknown function. Let y_1 and y_2 be solutions of this differential equation. Show that for any constant c the function $y = y_1 + c\, y_2$ is also a solution.

8. A spring-mass system satisfies the initial value problem:

$$x'' + 2x' + 2x = 0,\ x(0) = -0.1,\ x'(0) = 0.$$

   Find the solution $x(t)$. Find also the time when the mass first returns to the equilibrium position.

9. Write $(1+i)^{2001}$ in polar form with $0 \le \theta < 2\pi$.

10. Solve the following differential equations. If no conditions are specified, find the general solution.

    (a) $y' = -e^{x+y}$.

    (b) $xy' + (x+1)y = e^{-x}, \quad y(\ln 2) = 0$.

11. (a) Compute the Taylor series for $f(x) = 1/(1-x)$ about $x = 0$ and find its radius of convergence.

    (b) Compute the Taylor series for $f(x) = 1/(1+x^2)$ about $x = 0$ and find its radius of convergence.

    (c) Compute the Taylor series for $f(x) = \arctan x$ about $x = 0$ and find its radius of convergence.