# Math 8: Calculus in one and several variables
# Spring 2017 - Homework 1

Return date: Wednesday 04/05/17

**keywords:** *Taylor polynomials, remainder estimate, geometric series*

*Instructions:* Write your answers neatly and clearly on straight-edged paper, use complete sentences and label any diagrams. Please show your work; no credit is given for solutions without work or justification.

**exercise 1.** *(3 points)* Find the Taylor polynomial $T_3(x)$, for the function $f(x)$ at $a$.

a) $f(x) = 1 + x^2 + x^4$ at $a = 2$.
**Solution:** We have to find the Taylor polynomial $T_3(x)$ of degree 3 at $x = 2$. We can find the coefficients for the powers of $(x - 2)$ using a table:

| $k$th derivative | $f^{(k)}(x)$ | $f^{(k)}(2)$ | $\frac{f^{(k)}(2)}{k!}$ |
|---|---|---|---|
| 0: $f(x)$ | $1 + x^2 + x^4$ | 21 | 21 |
| 1: $f'(x)$ | $2x + 4x^3$ | 36 | 36 |
| 2: $f''(x)$ | $2 + 12x^2$ | 50 | $\frac{50}{2!} = 25$ |
| 3: $f^{(3)}(x)$ | $24x$ | 48 | $\frac{48}{3!} = 8$ |

Hence

$$\begin{aligned} T_3(x) &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f^{(3)}(2)}{3!}(x-2)^3 \\ &= 21 + 36(x-2) + 25(x-2)^2 + 8(x-2)^3. \end{aligned}$$

b) $f(x) = e^{x^2}$ at $a = 1$.
**Solution:** We have to find the Taylor polynomial $T_3(x)$ of degree 3 at $x = 1$. Again we can find the coefficients for the powers of $(x - 1)$ using a table:

| $k$th derivative | $f^{(k)}(x)$ | $f^{(k)}(1)$ | $\frac{f^{(k)}(1)}{k!}$ |
|---|---|---|---|
| 0: $f(x)$ | $e^{x^2}$ | $e^1 = e$ | $e$ |
| 1: $f'(x)$ | $2xe^{x^2}$ | $2e$ | $2e$ |
| 2: $f''(x)$ | $(4x^2 + 2)e^{x^2}$ | $6e$ | $\frac{6e}{2!} = 3e$ |
| 3: $f^{(3)}(x)$ | $(8x^3 + 12x)e^{x^2}$ | $20e$ | $\frac{20e}{3!} = \frac{10e}{3}$ |

Hence

$$\begin{aligned} T_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 \\ &= e + 2e(x-1) + 3e(x-1)^2 + \frac{10e}{3}(x-1)^3. \end{aligned}$$

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**exercise 2.** *(4 points)* For each of the following problems, write out enough terms of the 100th Taylor polynomial

$$T_{100}(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{100} x^{100}$$

for the function $f(x)$ at the point $a$, to make the pattern obvious. Then write down an explicit expression for $c_n$.

a) $f(x) = e^{2x}$ at $a = 0$.
**Solution:** We can make a table to find the pattern:

| $k$th derivative | $f^{(k)}(x)$ | $f^{(k)}(0)$ | $\frac{f^{(k)}(0)}{k!}$ |
|---|---|---|---|
| 0: $f(x)$ | $e^{2x}$ | $e^{2\cdot 0} = e^0 = 1$ | $1$ |
| 1: $f'(x)$ | $2e^{2x}$ | $2 \cdot 1 = 2$ | $2$ |
| 2: $f''(x)$ | $2^2 \cdot e^{2x}$ | $2^2$ | $\frac{2^2}{2!}$ |
| 3: $f^{(3)}(x)$ | $2^3 \cdot e^{2x}$ | $8$ | $\frac{2^3}{3!}$ |
| **n**: $f^{(n)}(x)$ | $2^n \cdot e^{2x}$ | $2^n$ | $\frac{2^n}{n!}$ |

Hence $c_n = \frac{f^{(n)}(0)}{n!} = \frac{2^n}{n!}$ and

$$T_n(x) = 1 + 2x + \frac{2^2}{2!}x^2 + \frac{2^3}{3!}x^3 + .... + \frac{2^n}{n!}x^n.$$

b) $f(x) = \ln(x+1)$ at $a = 0$.
**Solution:** Again we make a table to find the pattern:

| $k$th derivative | $f^{(k)}(x)$ | $f^{(k)}(0)$ | $\frac{f^{(k)}(0)}{k!}$ |
|---|---|---|---|
| 0: $f(x)$ | $\ln(x+1)$ | $\ln(1) = 0$ | $0$ |
| 1: $f'(x)$ | $\frac{1}{x+1}$ | $1$ | $1$ |
| 2: $f''(x)$ | $\frac{-1}{(x+1)^2}$ | $-1$ | $\frac{-1}{2!} = -\frac{1}{2}$ |
| 3: $f^{(3)}(x)$ | $\frac{(-1)\cdot(-2)}{(x+1)^3}$ | $1 \cdot 2$ | $\frac{2}{3!} = \frac{1}{3}$ |
| 4: $f^{(4)}(x)$ | $\frac{(-1)\cdot(-2)\cdot(-3)}{(x+1)^4}$ | $-1 \cdot 2 \cdot 3$ | $\frac{-1\cdot 2\cdot 3}{4!} = -\frac{1}{4}$ |
| **n**: $f^{(n)}(x)$ | $\frac{(-1)^{n+1}\cdot(n-1)!}{(x-1)^n}$ | $(-1)^{n+1} \cdot (n-1)!$ | $\frac{(-1)^{n+1}\cdot(n-1)!}{n!} = \frac{(-1)^{n+1}}{n}$ |

Hence $c_n = \frac{(-1)^{n+1}\cdot(n-1)!}{n!} = \frac{(-1)^{n+1}}{n}$ for all $n \geq 1$ and

$$T_n(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + .... + \frac{(-1)^{n+1}}{n}x^n.$$

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**exercise 3.** *(3 points)*

a) Find the Taylor polynomial $T_3(x)$, for the function

$$f(x) = x \cdot \ln(x) \quad \text{at the point} \quad a = 1.$$

**Solution:** Using the method from **exercise 1** and **2** we find that

$$T_3(x) = (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3,$$

where $f'(x) = \ln(x) + 1, f''(x) = \frac{1}{x}, f^{(3)}(x) = -\frac{1}{x^2}$.

b) For the values $0.8 \le x \le 1.2$ estimate the accuracy of the approximation using the remainder estimate

$$|R_3(x)| = |f(x) - T_3(x)|$$

in Taylor's inequality (**Theorem 11.10.9** of the book). Justify your answer.
**Solution:** Taylor's inequality states that if $|f^{(4)}(x)| \le M$ for all $x$ satisfying $|x-1| \le d$. Then

$$|R_3(x)| = |f(x) - T_3(x)| \le \frac{M}{4!} \cdot |x-1|^4$$

for all $x$, such that $|x-1| \le d$.
We are interested in the interval $0.8 \le x \le 1.2$ which is equivalent to $|x-1| \le 0.2$.
So it remains to estimate $|f^{(4)}(x)| = \left|\frac{2}{x^3}\right|$ in this interval. As $\frac{2}{x^3}$ is a monotonically decreasing function for $0.8 \le x \le 1.2$, its maximum is attained at $x = 0.8$ and

$$\left|\frac{2}{x^3}\right| = \frac{2}{x^3} \le \frac{2}{0.8^3} = M \quad \text{for all} \quad 0.8 \le x \le 1.2.$$

Hence

$$|R_3(x)| = |f(x) - T_3(x)| \le \frac{2}{4! \cdot 0.8^3} \cdot 0.2^4 \simeq 0.000260.$$

**exercise 4.** *(3 points)* Suppose we use the following estimate for $3\cos(x)$:

$$3\cos(x) \simeq 3 - \frac{3}{2}x^2.$$

a) Explain why it's okay to estimate the error using either $R_2(x)$ or $R_3(x)$. (Note that we get a better estimate using $R_3(x)$.)
**Solution:** We have

$$f'(x) = -3\sin(x), \; f''(x) = -3\cos(x), \; f^{(3)}(x) = 3\sin(x) \quad \text{and} \quad f^{(4)}(x) = 3\cos(x).$$

In this case we have that $T_3(x) = T_2(x)$ and additionally

$$|f^{(3)}(x)| = |3\sin(x)| \leq 3 = M_3 \quad \text{and} \quad |f^{(4)}(x)| = |3\cos(x)| \leq 3 = M_4 \quad \text{for all} \quad x \in \mathbb{R}\,.$$

With the above bounds we get for all $|x| \leq 1$ in Taylor's inequality

$$|R_2(x)| = |f(x) - T_2(x)| \leq \frac{3}{3!} \cdot |x|^3 \quad \text{and} \quad |R_3(x)| = |f(x) - T_2(x)| \leq \frac{3}{4!} \cdot |x|^4,$$

as $T_2(x) = T_3(x)$. The latter inequality is better for $|x| \leq 1$. Therefore we can use the second inequality for our error estimate.

b) Use the boxed statement on page 1 of the Error Estimates handout to get a bound on the error in computing $3\cos(0.1)$ using the polynomial above. Show your work.
**Solution:** From the Taylor inequality for $T_3(x)$ we get with $|x| \leq 0.1 \leq 1$:

$$|R_2(x)| = |R_3(x)| = |f(x) - T_2(x)| \leq \frac{3}{4!} \cdot |x|^4 \leq \frac{3}{4!} \cdot 0.1^4 = 0.0000125.$$

**Note:** Using the inequality for $R_2(x)$ we would only get $|R_2(x)| \leq 0.0005$.

**exercise 5.** *(3 points)* Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum:

a) $\sum_{n=0}^{\infty} \frac{5}{\pi^n}$.
**Solution:** We rewrite the sum as $5 \cdot \sum_{n=0}^{\infty} \left(\frac{1}{\pi}\right)^n$. As $\frac{1}{\pi} < 1$ this sum converges and by Ch. 11.2 **Theorem 4** of the book we have that

$$5 \cdot \sum_{n=0}^{\infty} \left(\frac{1}{\pi}\right)^n = \frac{5}{1 - \frac{1}{\pi}} = \frac{5\pi}{\pi - 1}.$$

b) $\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^n}$.
**Solution:** We rewrite the sum in the following way:

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^n} = 3 \cdot \sum_{n=0}^{\infty} \frac{3^n}{(-2)^n} = 3 \cdot \sum_{n=0}^{\infty} \left(\frac{3}{-2}\right)^n.$$

This series is divergent as $\left|\frac{3}{-2}\right| = \frac{3}{2} > 1$ and the geometric series is divergent for $|x| \geq 1$.

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**exercise 6.** *(4 points)* Find the values of $x$ for which the series converges. Find the sum of the series for those values of $x$.

a) $\sum_{n=1}^{\infty}(x+2)^n$.
**Solution 1:** We can use the ratio test (see Ch. 11.6, page 779): For $a_n = (x+2)^n$ we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(x+2)^n}\right| = |x+2|.$$

By the ratio test we conclude that the series converges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x+2| < 1$. This is the interval $-3 < x < -1$. The series diverges for $|x+2| > 1$. By plugging in the points $x = -3$ and $x = -1$ we see that the series diverges in these points.
**Solution 2:** Alternatively we could also argue that for $y = (x+2)$ this is the geometric series which is convergent for $|y| < 1$ which is equal to $|x+2| < 1$ and divergent for $|y| \geq 1$ which is equal to $|x+2| \geq 1$.
Finally, as this is a geometric series, we have with $y = x+2$:

$$\sum_{n=1}^{\infty}(x+2)^n = \sum_{n=0}^{\infty}(x+2)^n - 1 = \frac{1}{1-(x+2)} - 1 = \frac{-1}{x+1} - 1 = -\frac{x+2}{x+1}.$$

b) $\sum_{n=0}^{\infty}\frac{2^n}{x^n}$.
**Solution 1:** We can use again the ratio test. For $a_n = \left(\frac{2}{x}\right)^n$ we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{2}{x}\right)^{n+1}}{\left(\frac{2}{x}\right)^n}\right| = \left|\frac{2}{x}\right|.$$

By the ratio test we conclude that the series converges if $|\frac{2}{x}| < 1$ or $|x| > 2$. This is equal to $x \in (-\infty, -2)$ or $x \in (2, \infty)$. By the ratio test the series diverges for $|x| < 2$. By plugging in the points $x = -2$ and $x = 2$ we see that the series diverges in these points.
**Solution 2:** Alternatively we could also argue that for $y = \frac{2}{x}$ this is the geometric series which is convergent for $|y| < 1$ which is equal to $|x| > 2$ and divergent for $|y| \geq 1$ which is equal to $|x| \leq 2$.
Again, as this is a geometric series, we get

$$\sum_{n=0}^{\infty}\frac{2^n}{x^n} = \frac{1}{1-\frac{2}{x}} = \frac{x}{x-2}.$$

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