Math 8: Calculus in one and several variables
Spring 2017 - Homework 2

Return date: Wednesday 04/12/17

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keywords: *ratio test, differentiation and integration of power series, Taylor series*

Instructions: Write your answers neatly and clearly on straight-edged paper, use complete sentences and label any diagrams. Please show your work; no credit is given for solutions without work or justification. Be sure to staple your homework.

exercise 1. *(3 points)* Determine whether the following series are convergent or not.

a) $\displaystyle\sum_{n=2}^{\infty} \frac{n^3+2n}{5n^3+1}$.

Solution: For $a_n = \frac{n^3+2n}{5n^3+1}$ we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3+2n}{5n^3+1} = \lim_{n\to\infty} \frac{1+\frac{2}{n^2}}{5+\frac{1}{n^3}} = \frac{1}{5}.$$

Hence the series is divergent as $\lim_{n\to\infty} a_n \neq 0$.

Note: In this case the ratio test is inconclusive as $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

b) $\displaystyle\sum_{n=10}^{\infty} \frac{(-2)^n}{n^2}$.

Solution: For $a_n = \frac{(-2)^n}{n^2}$ and $a_{n+1} = \frac{(-2)^{n+1}}{(n+1)^2}$ we have by the ratio test

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|a_{n+1}\cdot\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}}{\cdot(n+1)^2}\cdot\frac{n^2}{(-2)^n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}}{(-2)^n}\right|\cdot\left|\frac{n^2}{(n+1)^2}\right| \\
&= \lim_{n\to\infty}\frac{2^{n+1}}{2^n}\cdot\frac{n^2}{(n+1)^2} = 2\cdot\lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 2 > 1,
\end{aligned}$$

as $\lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1$. Hence the series is divergent by the ratio test.

exercise 2. *(4 points)* Determine the radius of convergence for the following series.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{3^n}(x-4)^n$.

Solution: For $a_n = \frac{n^2}{3^n}(x-4)^n$ and $a_{n+1} = \frac{(n+1)^2}{3^{n+1}}(x-4)^{n+1}$ we have by the ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2\cdot(x-4)^{n+1}\cdot 3^n}{n^2\cdot(x-4)^n\cdot 3^{n+1}}\right| = \frac{|x-4|}{3}\cdot\lim_{n\to\infty}\frac{(n+1)^2}{n^2} = \frac{|x-4|}{3},$$

as $\lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1$. We obtain by the ratio test:
The series converges if $\frac{|x-4|}{3} < 1$ or equally $|x-4| < 3$.
The series diverges if $\frac{|x-4|}{3} > 1$ or equally $|x-4| > 3$.
Hence the radius of convergence R is $R = 3$ and the interval of convergence is the interval of width 3 around the point $x = 4$.
Note: We could also use the test for the converge of power series directly that says that for $c_n = \frac{n^2}{3^n}$ we have $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \frac{1}{3} = \frac{1}{R}$.

b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \cdot \sqrt{n}}(x-1)^{2n}$.

Solution 1: Again we can use the ratio test. We have that $a_n = \frac{(-1)^n}{2^n \cdot \sqrt{n}}(x-1)^{2n}$ and $a_{n+1} = \frac{(-1)^{n+1}}{2^{n+1} \cdot \sqrt{n+1}}(x-1)^{2(n+1)}$. We first simplify $\left|\frac{a_{n+1}}{a_n}\right|$:

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-1)^{n+1} \cdot (x-1)^{2(n+1)} \cdot 2^n \cdot \sqrt{n}}{(-1)^n \cdot (x-1)^{2n} \cdot 2^{n+1} \cdot \sqrt{n+1}}\right| \\ &= \left|\frac{2^n}{2^{n+1}}\right| \cdot \left|\frac{(x-1)^{2n+2}}{(x-1)^{2n}}\right| \cdot \left|\frac{\sqrt{n}}{\sqrt{n+1}}\right| = \frac{|x-1|^2}{2} \cdot \frac{\sqrt{n}}{\sqrt{n+1}}. \end{aligned}$$

Hence

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-1|^2}{2} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{|x-1|^2}{2} \cdot \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{|x-1|^2}{2},$$

as $\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 1$. We obtain by the ratio test:
The series converges if $\frac{|x-1|^2}{2} < 1$ or equally $|x-1| < \sqrt{2}$.
The series diverges if $\frac{|x-1|^2}{2} > 1$ or equally $|x-1| > \sqrt{2}$.
Hence the radius of convergence R is $R = \sqrt{2}$ and the interval of convergence is the interval of width $\sqrt{2}$ around the point $x = 1$.

Solution 2: We could also use the test for the convergence of a power series directly. However, here we have to be careful, as this is not a regular power series with powers of $|x-1|^n$.
We can bypass this problem by substituting $y = (x-1)^2$ and find the radius of convergence for $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \cdot \sqrt{n}} y^n$. Here we get for the radius of convergence R_y:
For $c_n = \frac{(-1)^n}{2^n \cdot \sqrt{n}}$ we have $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \frac{1}{2} = \frac{1}{R_y}$. Hence $R_y = 2$. Resubstituting we obtain:
The series converges if $|y| < 2$ or equally $|x-1|^2 < 2 \Leftrightarrow |x-1| < \sqrt{2}$.

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exercise 3. *(6 points)* By manipulating familiar power series (geometric series or series in **Table 1** on page 808), find power series representations centered at 0 for the following functions and determine the radius of convergence.
Do not use the ratio test to determine the radius of convergence. Instead use what you already know about the convergence properties of the series you are manipulating.

a) $f(x) = \frac{5}{1-4x^2}$.
Solution: We substitute with $y = 4x^2$, hence

$$\frac{5}{1-4x^2} = \frac{5}{1-y} = 5 \cdot \frac{1}{1-y} = 5\sum_{n=0}^{\infty} y^n = 5\sum_{n=0}^{\infty}(4x^2)^n = \sum_{n=0}^{\infty} 5 \cdot 4^n \cdot x^{2n}.$$

As $\sum_{n=0}^{\infty} y^n$ is the geometric series, we have: The series converges if $|y| < 1$ or equally $|4x^2| < 1 \Leftrightarrow |x| < \frac{1}{2}$. The series diverges if $|y| > 1$ or equally $|4x^2| > 1 \Leftrightarrow |x| > \frac{1}{2}$.
Hence the radius of convergence is $\frac{1}{2}$.

b) $f(x) = \frac{1}{(1+x)^2}$.
Solution:

$$\frac{1}{(1+x)^2} = \left(-\frac{1}{1+x}\right)' \quad \text{and} \quad -\frac{1}{1+x} = -\frac{1}{1-(-x)} = -\sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^{n+1}x^n.$$

Again the latter is a geometric series with radius of convergence equal to 1. Differentiation does not change the radius of convergence by Ch. 11.9 **Theorem 2** of the book. This theorem also states that we obtain the power series for $\frac{1}{(1+x)^2}$ by differentiating the power series of $-\frac{1}{1+x}$ term by term. Hence

$$\frac{1}{(1+x)^2} = \left(-\frac{1}{1+x}\right)' = \sum_{n=0}^{\infty}(-1)^{n+1}(x^n)' = \sum_{\mathbf{n=1}}^{\infty}(-1)^{n+1}nx^{n-1}.$$

As differentiation does not change the radius of convergence R we have $R = 1$.

c) $f(x) = x^2 \cdot \tan^{-1}(x^3)$.
Solution: We know the power series expansion for $\tan^{-1}(y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot y^{2n+1}$. Its radius of convergence is $R = 1$. Using the substitution $y = x^3$ we get:

$$x^2 \cdot \tan^{-1}(x^3) = x^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot (x^3)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot x^2 \cdot x^{6n+3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{6n+5}.$$

The factor x^2 does not change the radius of convergence of the series, as for a fixed x we just multiply by the square of x. We get by resubstitution:
The series converges if $|y| < 1$ or equally $|x|^3 < 1 \Leftrightarrow |x| < 1$.
The series diverges if $|y| > 1$ or equally $|x|^3 > 1 \Leftrightarrow |x| > 1$.
Hence the radius of convergence is 1.

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exercise 4. *(2 points)* Use the definition of Taylor and Maclaurin series to compute the terms up to degree 4 of the Maclaurin series for $f(x) = \frac{1}{(1+x)^2}$ and compare with your answer to **exercise 3b)**.

Solution: We have to find the Taylor polynomial $T_4(x)$ of degree 4 at $x = 0$. We can find the coefficients for the powers of x using a table:

kth derivative	$f^{(k)}(x)$	$f^{(k)}(0)$	$\frac{f^{(k)}(0)}{k!}$
0: $f(x)$	$\frac{1}{(1+x)^2}$	1	1
1: $f'(x)$	$\frac{-2}{(1+x)^3}$	-2	-2
2: $f''(x)$	$\frac{(-2)\cdot(-3)}{(1+x)^4}$	$2\cdot 3$	$\frac{2\cdot 3}{2!} = 3$
3: $f^{(3)}(x)$	$\frac{(-2)\cdot(-3)\cdot(-4)}{(1+x)^5}$	$-2\cdot 3\cdot 4$	$\frac{-2\cdot 3\cdot 4}{3!} = -4$
4: $f^{(4)}(x)$	$\frac{(-2)\cdot(-3)\cdot(-4)\cdot(-5)}{(1+x)^6}$	$2\cdot 3\cdot 4\cdot 5$	$\frac{2\cdot 3\cdot 4\cdot 5}{4!} = 5$

Hence

$$T_4(x) = 1 - 2x + 3x^2 - 4x^3 + 5x^4 = \sum_{\mathbf{n=1}}^{5} (-1)^{n+1} n x^{n-1} = \sum_{\mathbf{n=0}}^{4} (-1)^n (n+1)\cdot x^n.$$

Hence $T_4(x)$ is equal to the truncated power series $\sum_{\mathbf{n=1}}^{5} (-1)^{n+1} n x^{n-1}$ of $f(x)$ at $x = 0$.

exercise 5. *(3 points)* For the function $f(x) = \cos(x)$.

a) Find the Taylor series for $f(x)$ centered at $\frac{\pi}{2}$. Find the complete series, not just the first few terms.
Solution: We can make a table to find the pattern:

kth derivative	$f^{(k)}(x)$	$f^{(k)}(\frac{\pi}{2})$	$\frac{f^{(k)}(\frac{\pi}{2})}{k!}$
0: $f(x)$	$\cos(x)$	0	0
1: $f'(x)$	$-\sin(x)$	-1	-1
2: $f''(x)$	$-\cos(x)$	0	$\frac{0}{2!} = 0$
3: $f^{(3)}(x)$	$\sin(x)$	1	$\frac{1}{3!}$
4: $f^{(4)}(x)$	$\cos(x)$	0	$\frac{0}{4!} = 0$
5: $f^{(5)}(x)$	$-\sin(x)$	-1	$\frac{-1}{5!}$

We see that the pattern for the first two rows is repeating after four steps. The even derivatives $f^{(2n)}(\frac{\pi}{2})$ at $x = \frac{\pi}{2}$ are always 0. Hence

$$T_{2n+1}(x) = -(x - \frac{\pi}{2}) + \frac{1}{3!}(x - \frac{\pi}{2})^3 - \frac{1}{5!}(x - \frac{\pi}{2})^5 + ... + \frac{(-1)^{n+1}}{(2n+1)!}(x - \frac{\pi}{2})^{2n+1}.$$

Hence the Taylor series is $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x - \frac{\pi}{2})^{2n+1}$.

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b) Find an upper bound for $|R_n(x)| = |f(x) - T_n(x)|$, the remainder after the nth degree Taylor polynomial and check that

$$\lim_{n\to\infty} R_n(x) = 0 \text{ for every } x.$$

Solution: We first fix an interval $|x - \frac{\pi}{2}| \le d$ of width d around $x = \frac{\pi}{2}$. Taylor's inequality states that if $|f^{(n+1)}(x)| \le M$ for all x satisfying $|x - \frac{\pi}{2}| \le d$. Then

$$|R_n(x)| = |f(x) - T_n(x)| \le \frac{M}{(n+1)!} \cdot |x - \frac{\pi}{2}|^{n+1}$$

for all x, such that $|x - \frac{\pi}{2}| \le d$.
We first find M: From our table in a) we know that $|f^{(n+1)}(x)| = |\cos(x)| \le 1$ or $|f^{(n+1)}(x)| = |\sin(x)| \le 1$. Hence we can take $M = 1$. Then

$$|R_n(x)| \le \frac{1}{(n+1)!} \cdot |x - \frac{\pi}{2}|^{n+1} \le \frac{d^{n+1}}{(n+1)!}.$$

But for any fixed d we know by Ch. 11.10 inequality 10 (page 802) that

$$\lim_{n\to\infty} \frac{d^{n+1}}{(n+1)!} = \lim_{n\to\infty} \frac{d^n}{n!} \cdot \frac{d}{n+1} = 0.$$

Hence $\lim_{n\to\infty} |R_n(x)| = 0$ by the **Squeeze Theorem**. As d can be chosen arbitrarily we have convergence for all $x \in \mathbb{R}$.
Thus the Taylor series converges to the function $\cos(x)$ everywhere.

exercise 6. *(2 points)* Find the sums of the following series by associating them to a Taylor series. **Hint:** Look at **Table 1** on page 808 of the book.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{4n+1}}{n!}$.

Solution: We can modify the sum in the following way:

$$\sum_{n=0}^{\infty} \frac{x^{4n+1}}{n!} = \sum_{n=0}^{\infty} x \cdot \frac{(x^4)^n}{n!} = x \cdot \sum_{n=0}^{\infty} \frac{(x^4)^n}{n!} = x \cdot \sum_{n=0}^{\infty} \frac{y^n}{n!} = x \cdot e^y = x \cdot e^{x^4}.$$

Here we use the substitution $y = x^4$ and the fact that the Taylor series for e^y is $\sum_{n=0}^{\infty} \frac{y^n}{n!}$.

b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$.

Solution: By comparison with the power series on page 808 we see that this is the Maclaurin series for $\cos(x)$ at $x = 1$. Hence

$$\cos(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} = 1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} + ... + ...$$
